香 港 資 優 教 育 學 苑
The Hong Kong Academy for Gifted Education

International Mathematical Olympiad Preliminary Selection Contest – Hong Kong 2023

國際數學奧林匹克 — 香港選拔賽初賽 2023

20 May 2023 (Saturday)
2023 年 5 月 20 日（星期六）

Question Book
問題簿

Instructions to Contestants:

考生須知：

1. The contest comprises a 3-hour written test.
比賽以筆試形式進行，限時三小時。

2. Questions are bilingual. Answer all questions.
題目中英對照。全卷題目均須作答。

3. Put your answers on the answer sheet.
請將答案寫在答題紙上。

4. The use of calculators is NOT allowed.
不可使用計算機。

5. Measuring instruments like rulers, compasses, etc. can be used.
直尺、圓規及其它量度工具可作輔助之用。

Co-organised by The Hong Kong Academy for Gifted Education,
the Gifted Education Section of the Education Bureau and
International Mathematical Olympiad Hong Kong Committee
香港資優教育學苑、教育局資優教育組及國際數學奧林匹克香港委員會合辦

1. Someone extracted n consecutive digits in the infinite decimal representation of $\frac{1}{13}$ and found that the sum of the n digits is equal to 2023. Find the value of n. (1 mark)

某人在 $\frac{1}{13}$ 的無限小數表示式中選取了 n 個連續數字，並發現這 n 個數字之和為 2023。求 n 的值。（1 分）

2. Let m and n be positive integers such that $\sqrt{m}+\sqrt{n}=\sqrt{2023}$. Find the greatest possible value of $m+n$. (1 mark)

設 m、n 為正整數，使得 $\sqrt{m}+\sqrt{n}=\sqrt{2023}$。求 $m+n$ 的最大可能值。（1 分）

3. There is a 20×20 table and 400 cards printed with the numbers 1 to 400. The cards are then distributed to the cells of the table so that there is one card in each cell. After that, we put a red sticker on the card in each row with the largest number, and a blue sticker on the card in each column with the largest number. Let A denote the smallest number on the 20 cards with red stickers, and B denote the smallest number on the 20 cards with blue stickers. How many different possible values of $|A-B|$ are there? (1 mark)

現有一個 20×20 的表格和 400 張分別印上數字 1 至 400 的咭片，然後我們把這些咭片分配到表格的 400 格，使得每格有一張咭片。之後我們在每行數字最大的一張咭片貼上紅色貼紙，並在每列數字最大的一張咭片貼上藍色貼紙。在 20 張貼上紅色貼紙的咭片中，記當中最小的數字為 A；在 20 張貼上藍色貼紙的咭片中，記當中最小的數字為 B。那麼，$|A-B|$ 有多少個不同的可能值？（1 分）

4. A test question reads ‘write down three consecutive positive integers not exceeding 2023 in ascending order, such that one of them is a multiple of 6, another one is a multiple of 7 and the remaining one is a multiple of 8’. How many different correct answers are there for this question? (1 mark)

某次測驗的其中一道題是這樣的：「從小至大寫下三個不超過 2023 的連續正整數，使得當中其中一個數是 6 的倍數，另一個是 7 的倍數，餘下一個則是 8 的倍數」。這道題有多少個不同的正確答案？（1 分）

5. $ABCD$ is a square of side length 1. BC is extended to E and DC is extended to F such that $BE = DF = 3$. The circumcircle of ΔAEF meets the extensions of CB and CD at G and H respectively. Find GH. (1 mark)

$ABCD$ 是邊長為 1 的正方形。現把 BC 延長至 E，DC 延長至 F，使得 $BE = DF = 3$。ΔAEF 的外接圓與 CB 和 CD 的延線分別交於 G 和 H。求 GH。（1 分）

6. Let n be a positive integer such that $1^3+2^3+\cdots+n^3$ is divisible by $n+3$. Find the greatest possible value of n. (1 mark)

設 n 為正整數，使得 $1^3+2^3+\cdots+n^3$ 可被 $n+3$ 整除。求 n 的最大可能值。（1 分）

7. In ΔABC, $\angle BAC = 150°$ and $BC = 74$. D is a point on BC such that $BD = 14$ and $\angle ADB = 60°$. Find the area of ΔABC. (1 mark)

在 ΔABC 中，$\angle BAC = 150°$，且 $BC = 74$。D 是 BC 上的一點，使得 $BD = 14$，且 $\angle ADB = 60°$。求 ΔABC 的面積。（1 分）

8. How many solutions are there to the equation $\log_3 x = 3\sin(3\pi x)$? (1 mark)

方程 $\log_3 x = 3\sin(3\pi x)$ 有多少個解？ （1 分）

9. Find the largest real root to the equation $\sqrt[3]{x^3+3x^2-4} - x = \sqrt[3]{x^3-3x+2} - 1$. (1 mark)

求方程 $\sqrt[3]{x^3+3x^2-4} - x = \sqrt[3]{x^3-3x+2} - 1$ 的最大實根。 （1 分）

10. Find the remainder when $1\times2\times3\times4 + 2\times3\times4\times5 + \cdots + 2023\times2024\times2025\times2026$ is divided by 1000. (1 mark)

求 $1\times2\times3\times4 + 2\times3\times4\times5 + \cdots + 2023\times2024\times2025\times2026$ 除以 1000 時的餘數。 （1 分）

11. In a school there are 2023 students, numbered 1 to 2023. The teacher met the students one by one in the order of their numbers, and gave a candy to each student except if that would mean three students whose numbers form an arithmetic sequence all got candies. In this way, the teacher would give a candy to students 1 and 2, but not student 3 (as 1, 2, 3 form an arithmetic sequence), then to students 4 and 5, but not to students 6 and 7 (as both 4, 5, 6 and 1, 4, 7 are arithmetic sequences), and so on. How many students got candies in the end? (2 marks)

某校有 2023 名學生，分別編號為 1 至 2023。老師按學號的順序逐一與學生見面，並給每名學生一顆糖果，除非這樣會使得三名學號組成等差數列的學生均獲得糖果。如是者，1 號和 2 號學生獲得糖果，但 3 號學生不獲糖果（因為 1、2、3 組成等差數列），然後 4 號和 5 號學生獲得糖果，但 6 號和 7 號學生不獲糖果（因為 4、5、6 和 1、4、7 均組成等差數列），如此類推。那麼，最終有多少名學生會獲發糖果？ （2 分）

12. *ABCD* is a trapezium with *AD* // *BC*, *AD* > *BC* and *AB* = *CD*. *P* is a point on the plane whose distance from *A*, *B*, *C*, *D* are 1, 2, 3, 4 respectively. Find $AD : BC$. (2 marks)

ABCD 是梯形，其中 *AD* // *BC*，*AD* > *BC*，且 *AB* = *CD*。*P* 是平面上的一點，它與 *A*、*B*、*C*、*D* 的距離分別是 1、2、3、4。求 $AD : BC$。 （2 分）

13. In ΔABC, $AC = \sqrt{3}AB$ and $BC = 2$. D is a point inside ΔABC such that $\angle BDC = 90^\circ$, $\angle DAC = 18^\circ$ and $BD = 1$. Find $\angle DAB$. (2 marks)

在 ΔABC 中，$AC = \sqrt{3}AB$ 及 $BC = 2$。D 是 ΔABC 內的一點，使得 $\angle BDC = 90^\circ$、$\angle DAC = 18^\circ$ 及 $BD = 1$。求 $\angle DAB$。 （2 分）

14. In a country there are only four types of coins, of denominations 74, 87, 111 and 124 dollars respectively. In how many different ways can one pay exactly 2023 dollars using these coins? (2 marks)

某國家只有四種硬幣，面值分別為 74、87、111 和 124 元。利用這些硬幣，有多少種不同方法付款剛好 2023 元？ （2 分）

15. There is a 3×3 table and 9 cards printed with the numbers –4, –3, –2, –1, 0, 1, 2, 3, 4. How many ways are there for these cards to be distributed to the cells of the table so that there is one card in each cell, and that the sum of the numbers of the three cards on each row, each column and each of the two diagonals of the table is non-negative? (2 marks)

現有一個 3×3 的表格和 9 張分別印上數字 –4、–3、–2、–1、0、1、2、3、4 的咭片。有多少種方法可以把這些咭片分配到表格的 9 格，使得每格有一張咭片，且表格每行、每列和兩條對角線上三張咭片的數字之和均為非負？ （2 分）

16. In a team game, 12 players stand at the 12 vertices of a regular 12-sided polygon. Each player has a red flag and a blue flag, and then randomly put up one flag, all at the same time. If there are four players who put up flags of the same colour and whose positions form a rectangle, the team loses. Otherwise the team wins. What is the probability for the team to win? (2 marks)

在一個隊際遊戲中，12 人分別站在一個正十二邊形的 12 個頂點，每人均有一面紅旗和一面藍旗。然後，他們同時隨機選擇一面旗子舉起。如果當中有四人舉起相同顏色的旗子，且四人所處的位置成一長方形，則隊伍在遊戲中落敗，否則隊伍勝出。那麼，隊伍勝出的概率是多少？ （2 分）

17. *ABCD* is a square. *P* is a point inside *ABCD* such that $\angle APD + \angle BPC = 180°$ and $\angle BPC$ is acute. If $PB = 3$ and $PC = 4$, find *BC*. (2 marks)

ABCD 為正方形。*P* 是 *ABCD* 內的一點，使得 $\angle APD + \angle BPC = 180°$，且 $\angle BPC$ 為銳角。若 $PB = 3$，且 $PC = 4$，求 *BC*。 （2 分）

18. In a chess tournament, there were *n* contestants and any two of them played at most one game against each other. Each contestant has played exactly 100 games. For any two contestants *A* and *B* who had played against each other, there were exactly 50 other participants who have played against both *A* and *B*. On the other hand, for any two contestants *C* and *D* who had not played against each other, there were exactly 4 other participants who have played against both *C* and *D*. Find the value of *n*. (2 marks)

某次象棋比賽有 *n* 人參加，當中任意兩人之間對賽最多一次。每位選手均進行了剛好 100 場比賽。對任意兩位對賽過的選手 *A* 和 *B*，皆有剛好 50 名其他選手與 *A* 和 *B* 都對賽過。對任意兩位未曾對賽的選手 *C* 和 *D*，皆有剛好 4 名其他選手與 *C* 和 *D* 都對賽過。求 *n* 的值。 （2 分）

19*. There are 20 students in a class, numbered 1 to 20. They have taken a test with *n* questions. After the test, the teacher tried to pick some students to draft the solutions, and as such it is necessary that the picked students together can solve all questions. The teacher found that this can be only done by choosing all students whose class numbers are even, or all students whose class numbers are multiples of 5, or any combination that includes all students from one of the two groups mentioned above. Find the smallest possible value of *n*. (2 marks)

某班有 20 名學生，分別編號為 1 至 20。他們進行了一次測驗，當中共有 *n* 道題。測驗結束後，老師希望選若干名學生負責撰寫題解，因此需要確保選出的學生合共能答對所有試題。老師發現她只能選所有學號為偶數的學生，或是所有學號為 5 的倍數的學生，或是任何包括了上述兩組之一的所有學生的組合。求 *n* 的最小可能值。 （2 分）

20. Let *u*, *v*, *w* be positive integers, with *v* and *w* not exceeding 2023, such that $u - v = 2w$ and $(u^2 + w^2)(v^2 + w^2)$ is divisible by 289. How many sets of possible values of (u, v, w) are there? (2 marks)

設 *u*、*v*、*w* 為正整數，其中 *v* 和 *w* 均不超過 2023，使得 $u - v = 2w$，且 $(u^2 + w^2)(v^2 + w^2)$ 可被 289 整除。那麼，(u, v, w) 有多少組不同的可能值？ （2 分）

End of Paper

全卷完

* The wording of Question 19 has been revised from the live paper to fix some ambiguities.
由於原試題中有不清晰的地方，第 19 題的字眼已稍作修改。