**International Mathematical Olympiad**
**Hong Kong Preliminary Selection Contest 2022**

Outline of solutions

## Answers:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| **1.** | 89 | **2.** | 4044 | **3.** | 276 | **4.** | 607 |
| **5.** | $\dfrac{5\sqrt{39}}{2}$ | **6.** | 766 | **7.** | $2 \times 10^{19}$ | **8.** | $9+6\sqrt{3}$ |
| **9.** | $\dfrac{5-2\sqrt{2}}{6}$ | **10.** | $10\sqrt{2}$ | **11.** | $\dfrac{55}{2}$ | **12.** | $\dfrac{16}{9}$ |
| **13.** | 9500 | **14.** | 2358 | **15.** | 44 | **16.** | $324\sqrt{2}$ |
| **17.** | 14 | **18.** | 17 | **19.** | 16 | **20.** | 15 |

## Solutions:

1. We have

$$\left(\frac{x+y}{x-y}\right)^2 = \frac{x^2+y^2+2xy}{x^2+y^2-2xy} = \frac{(\frac{3961}{1980}+2)xy}{(\frac{3961}{1980}-2)xy} = 7921 = 89^2$$

and so the answer is 89. (Note that $\dfrac{x+y}{x-y} > 0$ as $x > y > 0$.)

2. Let $a = u^3$ and $b = v^3$. Then we have $u^4 - 2022u^2 + 2023 = 0$ and $v^4 - 2022v^2 + 2023 = 0$, so $u^2$ and $v^2$ are the two roots of $t^2 - 2022t + 2023 = 0$. In particular we have $u^2 + v^2 = 2022$. Note also that $p + q = u^3 + 3uv^2 + v^3 + 3u^2v = (u+v)^3$ and similarly $p - q = (u-v)^3$. It follows that

$$(p+q)^{2/3} + (p-q)^{2/3} = (u+v)^2 + (u-v)^2 = 2(u^2+v^2) = 4044.$$

3. Since $20^{22}$ is a square number, $m^2n$, and hence $n$, must also be a square number. Write $n = k^2$. Then the equation becomes $m^2k^2 = 20^{22}$, or $mk = 20^{11}$. Each positive factor of $20^{11}$ corresponds to a choice of $m$, which in turn corresponds to a solution $(m, n)$ to the original equation. The answer is thus equal to the number of positive factors of $20^{11} = 2^{22}5^{11}$, which is $(22+1)(11+1) = 276$.

**Remark.** For a positive integer $n$ with prime factorisation $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$, the number of positive factors of $n$ is given by

$$(a_1+1)(a_2+1)\cdots(a_k+1).$$

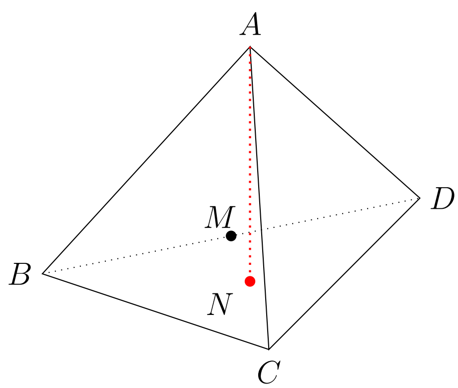
4. Let the digits in the box form the number $\overline{abc}$. Since 22! is divisible by $7 \times 11 \times 13 = 1001$ and $1000 \equiv -1 \pmod{1001}$, we have

$$\begin{aligned} 0 &\equiv \overline{1124000727777abc680000} \\ &= 1(1000)^7 + 124(1000)^6 + 727(1000)^4 + 777(1000)^3 + \overline{abc}(1000)^2 + 680(1000) \\ &\equiv -1 + 124 + 727 - 777 + \overline{abc} - 680 \\ &= \overline{abc} - 607 \qquad \pmod{1001} \end{aligned}$$

It follows that the answer is 607.

**Remark.** The technique in the solution provides a divisibility test for 1001, which works equally well for divisors of 1001 such as 7, 11 and 13.

5. Let the vertices of the pyramid be $A$, $B$, $C$, $D$, where $BD = 6$ and all other edges have length 5. Let also $M$ be the mid-point of $BD$ and $N$ be the foot of perpendicular from $A$ to the base $BCD$. Of course we have $BM = MD = 3$ and $CM = 4$.




Note that

$$AN^2 = AB^2 - BN^2 = AC^2 - CN^2 = AD^2 - DN^2.$$

As $AB = AC = AD = 5$, we have $BN = CN = DN$ and so $N$ is the circumcentre of $\triangle BCD$. The circumradius of $\triangle BCD$ is given by the extended sine formula

$$BN = \frac{CD}{2\sin B} = \frac{5}{2 \cdot \frac{4}{5}} = \frac{25}{8}.$$

The area of $\triangle BCD$ is $\frac{1}{2} \cdot BD \cdot MC = 12$. The volume of the pyramid is thus

$$\frac{1}{3} \cdot 12 \cdot AN = 4 \cdot \sqrt{5^2 - \left(\frac{25}{8}\right)^2} = \frac{5\sqrt{39}}{2}.$$

6. We have $n = 8^3 + 8^2P + 8Q + R = 11^2R + 11P + Q$ (where $P$, $Q$ and $R$ are at most 7), which simplifies to

$$120R = 53P + 7Q + 512. \tag{*}$$

Considering modulo 8 in (*), we get $Q \equiv 5P \pmod 8$, so $P$ and $Q$ are both odd (if $P$ is even then $Q - P \equiv 4P \equiv 0 \pmod 8$ which is impossible since $P$ and $Q$ are distinct and at most 7). This gives four possibilities of $(P, Q)$, namely, $(1,5)$, $(3,7)$, $(5,1)$ and $(7,3)$. Considering modulo 5 in (*), we get $2Q \equiv 2P + 3 \pmod 5$, and among the four possibilities above only $(1,5)$ and $(3,7)$ work. Using (*),

- $(P, Q) = (1,5)$ gives $R = 5$;
- $(P, Q) = (3,7)$ gives $R = 6$.

As $P$, $Q$, $R$ are distinct, the former is rejected and from the latter we get the answer $n = 11^2(6) + 11(3) + 7 = 766$.

7. Suppose the sequence of numbers shown on the calculator screen is

$$n \longrightarrow p \longrightarrow q \longrightarrow r \longrightarrow 0$$

with $p$, $q$, $r$ nonzero. Note that $r$ is at least 1 and so $q$ consists of at least one even digit, which means $q \geq 2$. Hence $p$ consists of at least two even digits and is no less than 20. It follows that $n$ is at least a 20-digit number, and the smallest such number is $2 \times 10^{19}$, which one easily verifies to be a possible value of $n$. (The sequence in this case will be $20...00 \to 20 \to 2 \to 1 \to 0$.)

8. Let $u = \sqrt{2x+3}$. Then the equation becomes $2x^2 + 3u^2 = 7xu$, or $(2x - u)(x - 3u) = 0$. Hence either $u = 2x$ or $x = 3u$.

- If $u = 2x$, we get $2x = \sqrt{2x+3}$. Squaring both sides gives $4x^2 = 2x + 3$, which leads to $x = \frac{1}{4}(1 \pm \sqrt{13})$.
- If $x = 3u$, we get $\frac{x}{3} = \sqrt{2x+3}$. Squaring both sides gives $\frac{1}{9}x^2 = 2x + 3$, which leads to $x = 9 \pm 6\sqrt{3}$.

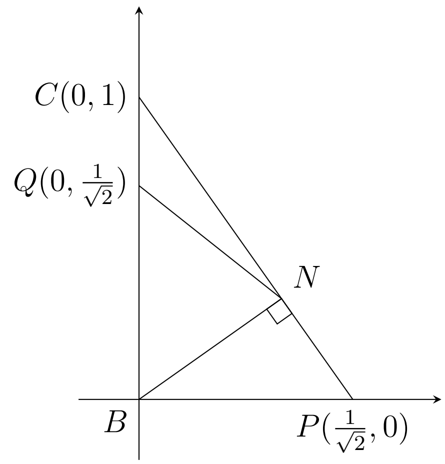
Among these, the largest one is $x = 9 + 6\sqrt{3}$. One can check that in this case

$$3u = \sqrt{18x + 27} = \sqrt{189 + 108\sqrt{3}} = \sqrt{\left(9 + \sqrt{108}\right)^2} = x$$

and so such $x$ satisfies the original equation, which is therefore the largest real root of the equation.

**Remark.** The final checking step is essential as the values of $x$ obtained are necessary but not sufficient conditions. Indeed one can check for instance that $x = 9 - 6\sqrt{3}$ is *not* a root to the original equation.

9. Set the square in the first quadrant of the coordinate plane with $B$ at the origin and $C$ at $(0,1)$. Then the coordinates of $P$ and $Q$ are $(\frac{1}{\sqrt{2}}, 0)$ and $(0, \frac{1}{\sqrt{2}})$ respectively:



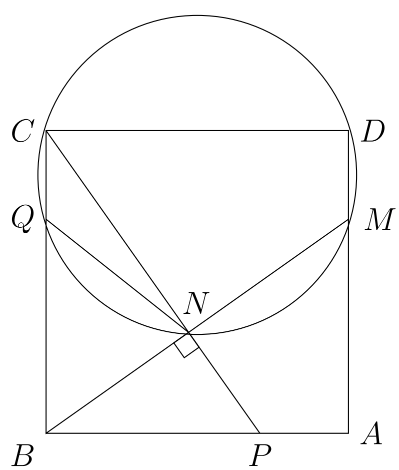

As $PC$ has slope $-\sqrt{2}$, the slope of $BN$ is $\frac{1}{\sqrt{2}}$. Let $N=(n,\frac{1}{\sqrt{2}}n)$. Considering the slope of $CN$, we have

$$\frac{\frac{1}{\sqrt{2}}n-1}{n}=-\sqrt{2},$$

which gives $n=\frac{1}{3}\sqrt{2}$. Hence $N=(\frac{1}{3}\sqrt{2},\frac{1}{3})$ and so

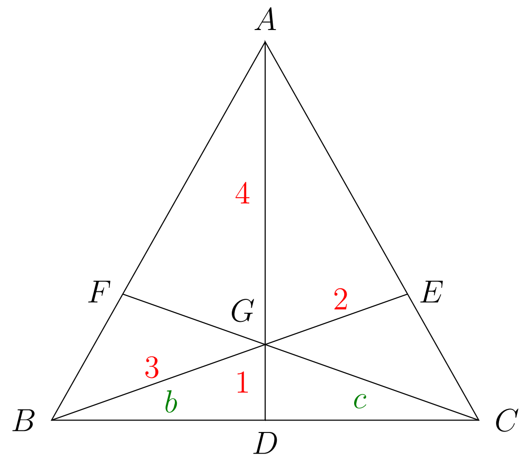
$$NQ^2=\left(\frac{1}{3}\sqrt{2}\right)^2+\left(\frac{1}{3}-\frac{1}{\sqrt{2}}\right)^2=\frac{5-2\sqrt{2}}{6}.$$

**Remark.** Many pure geometry solutions exist. For instance one can produce $BN$ to meet $AD$ at $M$. Noting that $\triangle BAM$ and $\triangle CBP$ are congruent, we have $AM=BP=BQ$. It follows that $CQMD$ is a rectangle, so $C$, $Q$, $N$, $M$, $D$ are concyclic due to the right angles:




With $DQ$ being a diameter of the circle, we have $\angle DNQ=90^\circ$ and so $\triangle DNQ$ is similar to $\triangle CNB$ (note that $\angle NDQ$ and $\angle NCB$ are equal as angles in the same segment). With the side lengths of $\triangle CNB$ as well as the length of $DQ$ easy to compute, we can use this pair of similar triangles to compute the length of $NQ$.

10. We use $[XYZ]$ to denote the area of $XYZ$. Suppose $[GDB]=b$ and $[GDC]=c$.

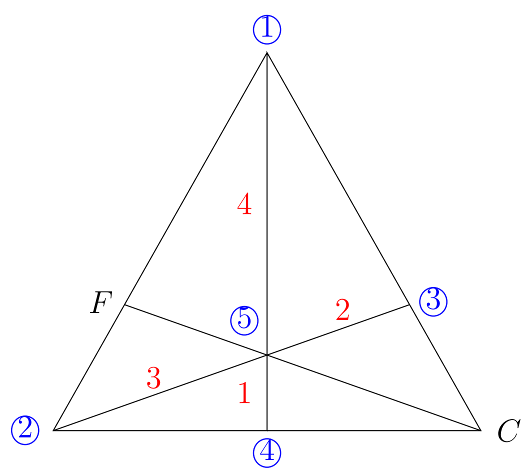



We now make use of the side length ratios to get the following:

- Using $AG : GD = 4 : 1$, we have $[AGB] = 4b$ and $[AGC] = 4c$.
- Using $BG : GE = 3 : 2$, we have $[AGE] = \frac{2}{3}(4b)$ and $[CGE] = \frac{2}{3}(b + c)$.
- As $[AGC] = [AGE] + [CGE]$, we have $4c = \frac{2}{3}(4b) + \frac{2}{3}(b + c)$, which gives $b = c$.
- We have $AF : FB = [AGC] : [BGC] = 4c : 2c = 2 : 1$, so $[FGB] = \frac{1}{3}[AGB] = \frac{4}{3}b$.
- It follows that $FG : GC = [FGB] : [CGB] = \frac{4}{3}b : 2b = 2 : 3$.

Since $CF = 5$, the last point above gives $GC = 3 = GB$. Hence $\triangle ABC$ must be isosceles with $AB = AC$ and hence $AD \perp BC$ (as $b = c$). It then follows that $BD = \sqrt{3^2 - 1^2} = 2\sqrt{2}$ and so the area of $\triangle ABC$ is $\frac{1}{2} \cdot BC \cdot AD = BD \cdot AD = 10\sqrt{2}$.

**Remark.** The solution can be simplified if one knows the *mass point* technique. With $AG : GD = 4 : 1$ and $BG : GE = 3 : 2$, we can assign masses of 1, 2, 3, 4, 5 at $A$, $B$, $E$, $D$, $G$ respectively:




Then the mass at $C$ would be 2 and the mass at $F$ would be 3, which gives $FG : GC = 2 : 3$ and from here we can proceed as in the original solution.

11. Let $x = \dfrac{1}{\sqrt{22}} \sec\theta$ and $y = \dfrac{1}{\sqrt{22}} \tan\theta$. Then $x^2 - y^2 = \dfrac{1}{22}$ and

$$\begin{aligned}
\frac{1-22xy}{x^2} &= \frac{1-\sec\theta\tan\theta}{\frac{1}{22}\sec^2\theta} \\
&= 22(\cos^2\theta - \sin\theta) \\
&= 22(-\sin^2\theta - \sin\theta + 1) \\
&= 22\left[\frac{5}{4} - \left(\sin\theta + \frac{1}{2}\right)^2\right]
\end{aligned}$$

The greatest possible value is thus $22 \cdot \frac{5}{4} = \frac{55}{2}$, attained when $\sin\theta = -\frac{1}{2}$ (which is feasible, e.g. with $\theta = -30^\circ$, $x = \frac{2}{\sqrt{66}}$ and $y = -\frac{1}{\sqrt{66}}$).

12. Note that

$$\begin{aligned}
S_0 + S_1 + S_2 + \cdots &= (1 + a + a^2 + \cdots) + (1 + b + b^2 + \cdots) + (1 + c + c^2 + \cdots) \\
&= \frac{1}{1-a} + \frac{1}{1-b} + \frac{1}{1-c}
\end{aligned}$$

As $a$, $b$, $c$ are roots of the equation $6x^3 + 5x^2 + 4x + 3 = 0$, the answer is equal to the sum of roots of the equation whose roots are $\frac{1}{1-a}$, $\frac{1}{1-b}$ and $\frac{1}{1-c}$. This equation is simply

$$6\left(1 - \frac{1}{x}\right)^3 + 5\left(1 - \frac{1}{x}\right)^2 + 4\left(1 - \frac{1}{x}\right) + 3 = 0,$$

or $6(x-1)^3 + 5x(x-1)^2 + 4x^2(x-1) + 3x^3 = 0$. The coefficient of $x^3$ is $6+5+4+3 = 18$ while the coefficient of $x^2$ is $-18 - 10 - 4 = -32$. The sum of roots is this $\frac{32}{18} = \frac{16}{9}$.

**Remarks.**

- One may also use $a+b+c = -\frac{5}{6}$, $ab+bc+ca = \frac{4}{6}$ and $abc = -\frac{3}{6}$ to directly compute

$$\frac{1}{1-a} + \frac{1}{1-b} + \frac{1}{1-c} = \frac{3 - 2(a+b+c) + (ab+bc+ca)}{1 - (a+b+c) + (ab+bc+ca) - abc} = \frac{16}{9}.$$

- One may check that $a$, $b$, $c$ all have modulus less than 1 (one of them is real and the other two are complex numbers), and so the infinite sum converges.

13. The answer is $\binom{50}{3}$ minus the number of choices of three rods that violate the triangle inequality. The number to be subtracted is the size of the set

$$S = \{(a,b,c) : 1 \leq a < b < c \leq 99,\ a+b < c,\ a,\ b,\ c \text{ are odd numbers}\}.$$

Note that for $(a,b,c) \in S$, $a+b$ is an even number between 4 and 98 (inclusive). For each such even number, we can easy count the number of such pairs $(a,b)$, and for each such $(a,b)$ we can easily count the number of choices of $c$ for which $(a,b,c) \in S$.

As an example, if we fix $a+b=20$, there would be 5 choices for $(a,b)$, namely, $(1,19)$, $(3,17)$, $(5,15)$, $(7,13)$ and $(9,11)$. For each of these 5 pairs of $(a,b)$, there are 40 choices of $c$ for which $(a,b,c)\in S$, namely, 21, 23, 25, ..., 97, 99.

In the same way, we can see that for each even number $2k$ where $k\in\{2,3,4,...,48,49\}$, there would be $\lfloor \frac{1}{2}k\rfloor$ choices of $(a,b)$ for which $a+b=2k$, and for each such $(a,b)$ there would be $50-k$ choices of $c$ for which $(a,b,c)\in S$. This can be summarised by the following table:

| Value of $k$ | Value of $a+b$ | No. of choices of $(a,b)$ | No. of choices of $c$ |
|---|---|---|---|
| 2 | 4 | $\lfloor\frac{2}{2}\rfloor=1$ | $50-2=48$ |
| 3 | 6 | $\lfloor\frac{3}{2}\rfloor=1$ | $50-3=47$ |
| 4 | 8 | $\lfloor\frac{4}{2}\rfloor=2$ | $50-4=46$ |
| 5 | 10 | $\lfloor\frac{5}{2}\rfloor=2$ | $50-5=45$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| 48 | 96 | $\lfloor\frac{48}{2}\rfloor=24$ | $50-48=2$ |
| 49 | 98 | $\lfloor\frac{49}{2}\rfloor=24$ | $50-49=1$ |

From the table, we see that

$$\begin{aligned}|S| &= 1\times 48+1\times 47+2\times 46+2\times 45+\cdots+24\times 2+24\times 1\\ &= 1\times 95+2\times 91+\cdots+24\times 3\\ &= \sum_{i=1}^{24} i(99-4i)\\ &= 99(1+2+\cdots+24)-4(1^2+2^2+\cdots+24^2)\\ &= 99\cdot\frac{24\cdot 25}{2}-4\cdot\frac{24(24+1)(2\cdot 24+1)}{6}\\ &= 10100\end{aligned}$$

and so the answer is

$$\binom{50}{3}-10100=\frac{50\cdot 49\cdot 48}{6}-10100=9500.$$

14. Note that $m$ is of the form ABCBA with A non-zero, so $n$ is of the form BCBA, ACBA, ABBA, ABCA or ABCB. The second, third and fourth cases can be combined, so we are down to three types of possible values of $n$:

- Type I — Thousands digit and tens digit equal, plus unit digit non-zero
- Type II — Thousands digit and unit digit equal
- Type III — Hundreds digit and unit digit equal

There are $9\times 10\times 9=810$ possibilities of Type I (9 choices for the common thousands and tens digit, 10 choices for the hundreds digit and 9 choices for the unit digit), and similarly $9\times 10\times 10=900$ possibilities for each of Type II and Type III. However,

- 90 numbers are of both Types I and II (those of the form XYXX with X non-zero);
- 81 numbers are of both Types I and III (those of the form XYXY with both X and Y non-zero);
- 90 numbers are of both Types II and III (those of the form XXYX with X non-zero);
- 9 numbers are of all three types (1111, 2222, ..., 9999).

By the inclusion-exclusion principle, the answer is $810+900+900-90-81-90+9 = 2358$.

15. For clearer illustration, we add a comma between two integers as we form $n$ by concatenating the integers from 1 to 100000, i.e. we write

$$n = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, ..., 99998, 99999, 100000.$$

We count the number of occurrence of 2022 according to the positions of the commas within these digits:

- Case 1: 2022 (i.e. no comma within the digits)

  The first 2 occurs either as a ten thousands digit or as a thousands digit. Each occurs 10 times, giving 20 occurrences in this case:

  - 20220,20221,20222,20223,20224,20225,20226,20227,20228,20229,...
  - 2021,2022,2023; 12021,12022,12023; ...; 92021,92022,92023.

- Case 2: 202,2

  The first part 202 may come from the three digit number 202, the four digit number 2202, or a five-digit number starting with 2 (there are 10 such numbers), giving 12 occurrences in this case:

  - 202,203
  - 2202,2203
  - 20202,20203; 21202,21203; ...; 29202,29203

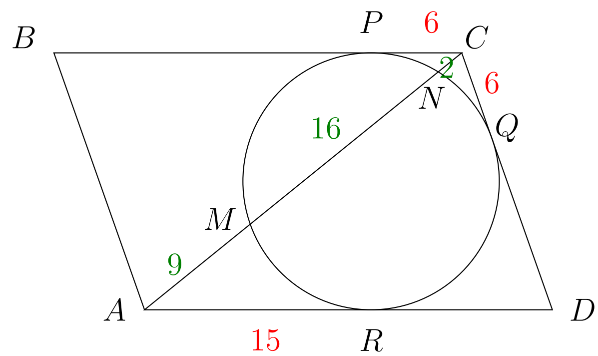
- Case 3: 20,22

  The first part 20 may come from the three-digit number 220, four-digit number 2220, or a five-digit number starting with 22 (there are 10 such numbers), giving 12 occurrences in this case:

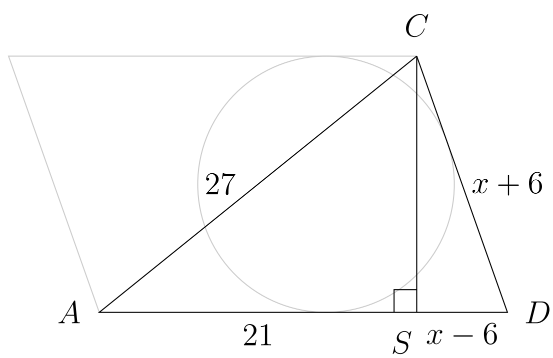
  - 220,221
  - 2220,2221
  - 22020,22021; 22120,22121; ...; 22920,22921

It follows that the answer is $20 + 12 + 12 = 44$.

16. Let $P$, $Q$, $R$ be the points where the circle touches $BC$, $CD$ and $DA$ respectively. By the power chord theorem (see Remark), we have $AR = \sqrt{AM \cdot AN} = 15$ and $CP = CQ = \sqrt{CN \cdot CM} = 6$.




Let $DR = DQ = x$ and $S$ be the foot of the perpendicular from $C$ to $AD$. Then we have the following lengths:

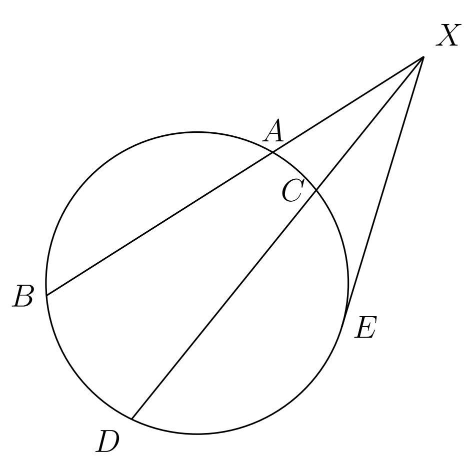



Using

$$27^2 - 21^2 = CS^2 = (x+6)^2 - (x-6)^2,$$

we get $x = 12$ and $CS = 12\sqrt{2}$. The area of $ABCD$ is this equal to $AD \cdot CS = (21+6) \cdot 12\sqrt{2} = 324\sqrt{2}$.

**Remark.** Let $X$ be a point outside a circle. The *power chord theorem* asserts that for any straight line passing through $X$ and intersecting the circle at $A$ and $B$ (possibly $A = B$ if the line is tangent to the circle), the value of $XA \cdot XB$ is a constant. Equivalently, we have $XA \cdot XB = XC \cdot XD = XE^2$ in the figure below. The theorem can be easily proved using the fact that $\triangle XAC \sim \triangle XDB$ and $\triangle XEA \sim \triangle XBE$.

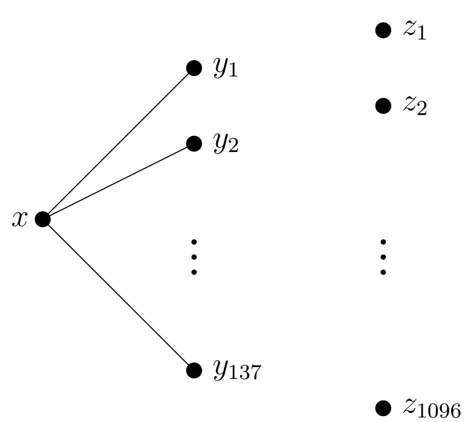


17. First note that there are exactly three even numbers and one odd number. (If all four numbers are even then all H.C.F.'s would be even, whereas if there are at most two even numbers then at most one H.C.F. can be even.) Suppose $a$ is the odd number and $b$, $c$, $d$ are the even numbers. We use $(m, n)$ to denote the H.C.F. of $m$ and $n$. Then $(a, b)$, $(a, c)$ and $(a, d)$ must be 1, 3 and 5 in some order. In particular, we note that $a$ is divisible by 3 and 5, whereas exactly one of $b$, $c$, $d$ is divisible by 3, and exactly one of $b$, $c$, $d$ is divisible by 5.

    Without loss of generality suppose $(b, c) = 2$, $(b, d) = 4$ and $(c, d) = k > 5$. Then $b$ and $d$ are both divisible by 4, and $c$ is divisible by 2 but not 4. It follows that $k$ is also divisible by 2 but not 4. Furthermore, $k$ is not divisible by 3 since $c$ and $d$ cannot be both divisible by 3, and similarly $k$ is not divisible by 5.

    Now we know that $k$ is greater than 5, is even, and is not divisible by 3, 4, 5. The smallest such $k$ is thus 14. Indeed this works; for instance if the four numbers are $(a, b, c, d) = (15, 4, 70, 84)$, then the H.C.F.'s involving $a$ will be equal to 1, 3, 5, while those not involving $a$ will be equal to 2, 4, 14. It follows that the answer is 14.

18. Consider any participant $x$. He has shaken hands with 137 other participants, say $y_1$, $y_2$, ..., $y_{137}$ (collectively known as Group $Y$ participants). Also, there are $1234-1-137 = 1096$ participants who have not shaken hands with $x$; let's call them $z_1$, $z_2$, ..., $z_{1096}$ (collectively known as Group $Z$ participants).

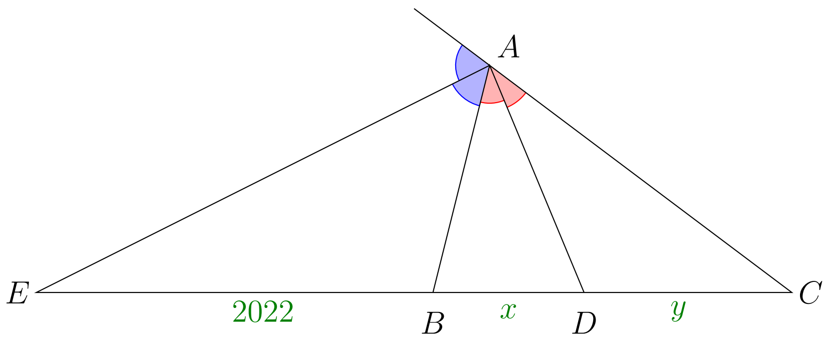
    
    

    We count the number of times a Group $Y$ participant has shaken hand with a Group $Z$ participant. This can be done in two ways:

    - Since no two participants in Group $Y$ can shake hands (otherwise together with $x$ there will be three participants shaking hands with each other), every Group $Y$ participant must have shaken hands with exactly 136 participants in Group $Z$. The total number of such handshakes is thus equal to $137 \times 136$.
    - On the other hand, every participant in Group $Z$ (say $z$) must shake hands with exactly $k$ participants in Group $Y$, because $x$ and $z$ have not shaken hands implies there are exactly $k$ participants (who must be in Group $Y$) who have shaken hands with both $x$ and $z$. It follows that the total number of such handshakes is also equal to $1096k$.

From the above two ways of counting we obtain the equality $1096k = 137 \times 136$, which gives $k = 17$.

19. Let the lengths of $BD$ and $DC$ be $x$ and $y$ respectively.




By the angle bisector theorem (see Remark), we have $\dfrac{DB}{DC} = \dfrac{AB}{AC} = \dfrac{EB}{EC}$, i.e.

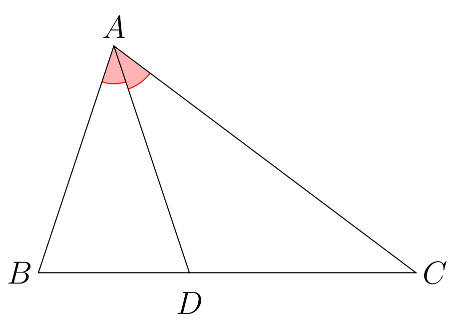
$$\frac{x}{y} = \frac{2022}{2022 + x + y}.$$

Making $y$ the subject, we have

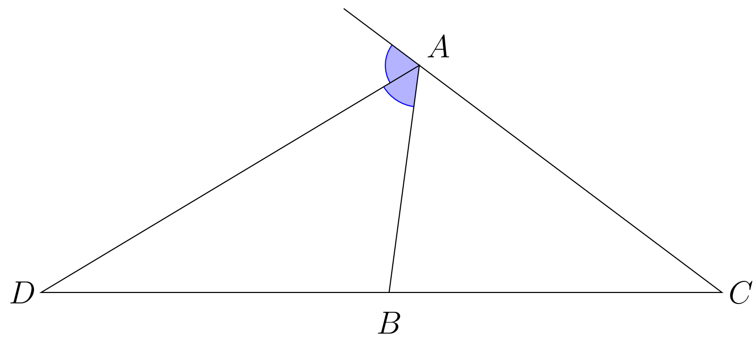
$$y = \frac{x^2 + 2022x}{2022 - x} = -x - 4044 + \frac{2022 \cdot 4044}{2022 - x}.$$

Hence we must choose positive integers $x$ for which $2022 - x$ is a positive integer dividing $2022 \cdot 4044$ (note that every such $x$ will result in a positive integer value of $y$ and the existence of such a figure satisfying all the conditions). As $2022 \cdot 4044 = 2^3 \cdot 3^2 \cdot 337^2$, its only positive factors that are less than 2022 are the 12 positive factors of $2^3 \cdot 3^2$ as well as 337, $337 \cdot 2$, $337 \cdot 3$ and $337 \cdot 4$. Hence there are altogether 16 such values of $2022 - x$, and they correspond to 16 possible values of $x$.

**Remark.** The *angle bisector theorem* asserts that the internal bisector of an angle of a triangle divides the opposite side in a ratio proportional to the lengths of the other two sides, i.e. in the figure below we have $BD : DC = AB : AC$.




The theorem can be proved by using the sine formula in $\triangle ADB$ and $\triangle ADC$, or by considering a point $E$ on the extension of $AD$ such that $AB$ and $CE$ are parallel (and then making use of similar triangles). There is also an external version, which says essentially the same thing except that we are considering the bisector of an external angle meeting the extension of the opposite side, dividing it in the same (external) ratio. More precisely, it says that in the figure below we again have $BD : DC = AB : AC$.

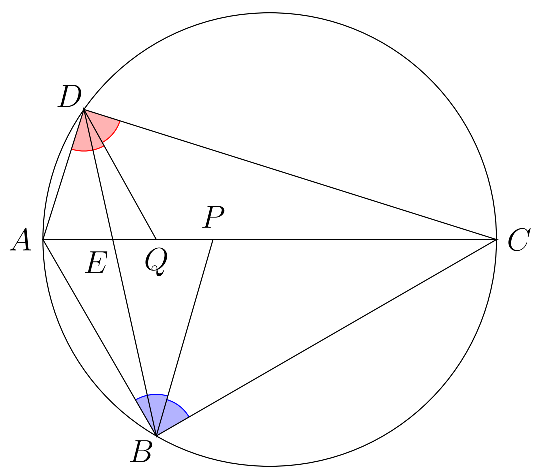



The proof can be obtained by slightly modifying a proof of the internal version of the theorem.

20. Note that $\sin\angle DAB = \sin\angle DCB$ as the two angles are supplementary. Using $[XYZ]$ to denote the area of $XYZ$, we have

$$\frac{AE}{EC} = \frac{[DAB]}{[DCB]} = \frac{\frac{1}{2}\cdot AD\cdot AB}{\frac{1}{2}\cdot DC\cdot BC} = \frac{AB}{BC}\cdot\frac{AD}{DC} \tag{*}$$

and the figure looks like this:




By the angle bisector theorem (see Remark to the previous question), we have

$$\frac{AB}{BC} = \frac{AP}{PC} \quad\text{and}\quad \frac{AD}{DC} = \frac{AQ}{QC}.$$

It thus follows from (*) that

$$\frac{AE}{EC} = \frac{AP}{PC}\cdot\frac{AQ}{QC}.$$

Setting $PC = x$ and using the given side lengths, this becomes

$$\frac{4}{5+x} = \frac{9}{x}\cdot\frac{6}{3+x}.$$

Solving gives $x = 15$ or $x = -\frac{9}{2}$, and of course the latter is rejected.