International Mathematical Olympiad

Preliminary Selection Contest 2003 - Hong Kong

Answers and Suggested Solutions

Answers:

1. 2	**2.** 35	**3.** 156	**4.** 4015014
5. 4005	**6.** $\frac{135}{49}\sqrt{3}$	**7.** 12	**8.** 2346
9. –2004	**10.** $\frac{3}{4}$	**11.** 6	**12.** 75
13. 13	**14.** 338350	**15.** $2\sqrt{21}$	**16.** $8\sqrt{3}-6$
17. $2\sqrt{3}$	**18.** 10	**19.** 134	**20.** $\frac{5}{7}$

Suggested Solutions:

1. Let $y=\sqrt[3]{2}$. Then $y^3=2$ and so $\left(1+\frac{1}{x}\right)^3=\left(1+\frac{1}{y^2+y+1}\right)^3=\left(1+\frac{y-1}{y^3-1}\right)^3=(1+y-1)^3=y^3=2$.

2. Every day $C_2^3=3$ pairs of students are on duty. Altogether $C_2^{15}=105$ pairs of students have been on duty. Therefore the answer is $105\div 3=35$.

3. Suppose n and $n+1$ are to be added, and write n in the form 1ABC. If any one of A, B and C is equal to 5, 6, 7 or 8, then carrying is needed in the addition. If B is 9 and C is not 9, or if A is 9 and B and C are not both 9, carrying is also required. Therefore, if no carrying is required, n must be of one of the following forms:

   1ABC   1AB9   1A99   1999

   where each of A, B, C is from 0 to 4. It follows that the answer is $5^3+5^2+5+1=156$.

4. Let n be a positive integer. Consider the numbers $4n$, $4n+1$, $4n+2$ and $4n+3$. The last two digits of their binary representations are 00, 01, 10 and 11 respectively. Among these, either $4n$ and $4n+3$ are magic numbers, or $4n+1$ and $4n+2$ are magic numbers. In any case, the sum of the two magic numbers among the four is equal to $8n+3$. Now the first magic number is $3=4(0)+3$, so the answer is $3+8(1+2+\cdots+1001)+3(1001)=4015014$.

5. First we count the number of 'increasing' integers of the form ABCA. In this case C must be greater than C. For each choice of A (there are 9 choices, namely, 1, 2, …, 9), there are $C_2^{10}=45$ choices for B and C (as we require C to be greater than B and we can choose among 0 to 9). So there are $9\times 45=405$ 'increasing' numbers of this form.

Next we count the number of 'increasing' integers of the form ABCD, where A is different from D. In this case we need only require D to be greater than A, whereas B and C can be arbitrary. Clearly, A and D have to be non-zero. So there are $C_2^9=36$ ways to choose A and D, and $10^2=100$ ways to choose B and C. Thus there are $36\times 100=3600$ 'increasing' numbers of this form.

It follows that the answer is $405+3600=4005$.

Alternative Solution

Clearly, an increasing number cannot end with 0. Consider all four-digit positive integers that does not end with 0. There are $9\times 10\times 10\times 9=8100$ such numbers. Among these, $9\times 10=90$ of them are palindromes (as all palindromes are of the form ABBA where A is from 1 to 9 while B is from 0 to 9). So $8100-90=8010$ of them are not palindromes. By grouping an integer with its reverse (i.e. ABCD with DCBA), we get $8010\div 2=4005$ pairs. In each pair, exactly one integer is increasing. So the answer is 4005.

6. Let ABC denote the triangle, and the lengths of the sides opposite A, B, C be $3k$, $5k$, $7k$ respectively. By cosine law, we have

$$\cos C=\frac{(3k)^2+(5k)^2-(7k)^2}{2(3k)(5k)}=-\frac{1}{2}$$

Hence C = 120°. By sine law,

$$\frac{7k}{\sin 120^\circ}=2\left(2\sqrt{3}\right)$$

from which we get $k=\frac{6}{7}$. It follows that the area of ΔABC is equal to

$$\frac{1}{2}(3k)(5k)\sin 120^\circ=\frac{15}{2}\left(\frac{6}{7}\right)^2\left(\frac{\sqrt{3}}{2}\right)=\frac{135}{49}\sqrt{3}.$$

7. If there are 729 apples, it is possible that 12 farmers took part in the sharing if the odd-numbered farmers took one-third of the remaining apples and the even-numbered farmers took one-half of the remaining apples. In this case one apple is left after the last farmer took his apple and the number of apples remaining is an integer throughout.

Now suppose p farmers took one-half of the apples and q farmers took one-third of the apples. We will show that $p+q \le 12$, so that 12 is indeed the greatest possible number of farmers. We have

$$1000 \ge 2^p \left(\frac{3}{2}\right)^q = 2^{p-q}3^q = 4^{\frac{p-q}{2}}3^q \ge 3^{\frac{p-q}{2}}3^q = 3^{\frac{p+q}{2}},$$

from which $p+q \le 12$ follows.

8. We have $0.9b < a < 0.91b$, so $1.9b < a+b < 1.91b$. Therefore we get the inequalities

$$1.9b < 99 \text{ and } 90 < 1.91b.$$

It follows that $48 \le b \le 52$. Putting each value of b into the inequality $0.9b < a < 0.91b$, we get two possibilities (a, b) = (46, 51) or (47, 52). In the latter case, the requirement $a+b < 99$ fails. Thus the former case gives the answer $46 \times 51 = 2346$.

9. Setting $y = x+1$, the equation becomes $\left(1+\frac{1}{y-1}\right)^y = \left(1+\frac{1}{2003}\right)^{2003}$. Setting $z = -y$, we get

$$\left(1+\frac{1}{-z-1}\right)^{-z} = \left(1+\frac{1}{2003}\right)^{2003}$$

$$\left(\frac{z}{z+1}\right)^{-z} = \left(1+\frac{1}{2003}\right)^{2003}$$

$$\left(\frac{z+1}{z}\right)^{z} = \left(1+\frac{1}{2003}\right)^{2003} \qquad \cdots\cdots(*)$$

Hence one possible integer solution is $z = 2003$, which corresponds to $y = -2003$ and $x = -2004$.

[From the above solution we also see that the solution for *z*, and hence the solution for *x*, is unique. Indeed, the left hand side of (*) is strictly increasing in *z* on the natural numbers, so there is no positive integer solution for *z* other than 2003. It is clear that *z* is not equal to 0 or –1. If *z* is smaller than –1, then *x* is a positive integer. Looking at the original equation, we readily see that this is not possible because on one hand *x* has to be divisible by 2003, and on the other hand the power $x+1$ has to be equal to 2003 by counting the exponent of 2003.]

10. On the unit circle, the points at –114°, –42°, 30°, 102° and 174° form a regular pentagon. So the sum of the cosines of these angles is 0. It follows that

$$\begin{aligned}&(\cos 42° + \cos 102° + \cos 114° + \cos 174°)^2 \\ &= [\cos(-114°) + \cos(-42°) + \cos 102° + \cos 174°]^2 \\ &= (-\cos 30°)^2 \\ &= \frac{3}{4}\end{aligned}$$

<u>Alternative Solution</u>

We have

$$\begin{aligned}&(\cos 42° + \cos 102° + \cos 114° + \cos 174°)^2 \\ &= (2\cos 72° \cos 30° + 2\cos 144° \cos 30°)^2 \\ &= 3(\cos 72° + \cos 144°)^2\end{aligned}$$

Now

$$\begin{aligned}&2\sin 36°(\cos 72° + \cos 144°) \\ &= 2\sin 36° \cos 72° + 2\sin 36° \cos 144° \\ &= \sin 108° - \sin 36° + \sin 180° - \sin 108° \\ &= -\sin 36°\end{aligned}$$

so that $\cos 72° + \cos 144° = -\frac{1}{2}$. It follows that the answer is $\frac{3}{4}$.

[As one could expect, there are many other ways of obtaining the answer, notably by solving cos 36° in surd form. The above solutions are surd-free in the entire process.]

11. First, observe that at least 6 unions must be formed. If a country joins only 2 unions, then it is in a common union with at most $49 \times 2 = 98$ countries. So each country must join at least 3 unions, and thus the smallest possible number of unions is $100 \times 3 \div 50 = 6$.

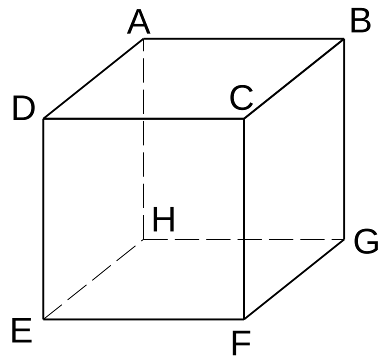
Next, 6 unions are enough. We can divide the 100 countries into 4 groups of 25 countries each. Choose 2 of the 4 groups at a time, and put the 50 countries in the 2 groups into a union. This gives $C_2^4 = 6$ unions. It is easy to see that in this way, any two countries belong to some union in common. It follows that the answer is 6.

12. Note that 7, 9, 31, …, 1999 form an arithmetic sequence with 167 terms and common difference 12. Since 12 and 25 are relatively prime, at least one of the terms is divisible by 25.

It remains to consider the product (mod 4). Note that each term is congruent to –1 (mod 4). As there are 167 terms, the product is also congruent to –1 (mod 4). Since it is divisible by 25, the answer must be 75.

13. Let a, b, c, d, e, f, g, h denote the numbers assigned, and A, B, C, D, E, F, G, H the average obtained, at the vertices A to H respectively. First we note that

$a+b+c+d+e+f+g+h=1+2+3+4+5+6+7+8=36$.



Also,

$$c+f=36-(b+d+h)-(a+e+g)=36-3(A+H).$$

Similarly, we have

$$e+f=36-3(A+B) \text{ and } f+g=36-3(A+D).$$

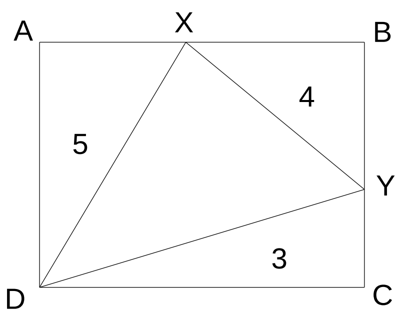
Therefore

$$\begin{aligned}3f &= (c+f)+(e+f)+(f+g)-(c+e+g)\\ &= [36-3(1+8)]+[36-3(1+2)]+[36-3(1+4)]-6\times 3\\ &= 39\end{aligned}$$

from which we get $f = 13$.

14. Fix a vertex A and draw a diameter from A. This diameter splits the remaining 200 vertices into two groups of 100 each. Number them as $P_1, P_2, \ldots, P_{100}$ and $Q_1, Q_2, \ldots, Q_{100}$ according to the distance of each vertex from A. Note that a triangle with A as vertex contains point C if and only if it is of the form AP_iQ_j where $i+j>100$. Hence there are $1+2+\cdots+100=5050$ triangles with A as a vertex which contains C. Since A is arbitrary we have $5050\times 201=1015050$ such triangles. However, in this way each triangle is counted three times. So the answer is $1015050\div 3=338350$.

15. Let $AX=a$ and $XB=b$. Then $AD=\frac{10}{a}$ and $BY=\frac{8}{b}$. Now the area of the rectangle is $\frac{10}{a}(a+b)=10+10R$, where $R=\frac{b}{a}$ is to be determined. The area of ΔDXY is then equal to $10+10R-5-4-3=10R-2$.



Consider ΔDYC. We have

$$6=DC\times CY=(a+b)\left(\frac{10}{a}-\frac{8}{b}\right)=2+10R-\frac{8}{R}.$$

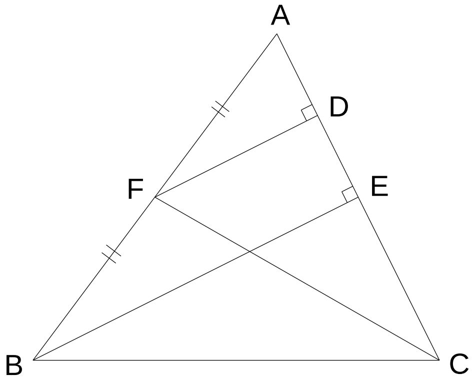
It follows that $5R^2-2R-4=0$. This gives $R=\frac{2\pm\sqrt{2^2-4(5)(-4)}}{2(5)}$. We take the positive root since R is positive. The answer is $10R-2=2\sqrt{21}$.

16. Let D be the foot of the perpendicular from F to AC. Note that $EC = \sqrt{BC^2 - BE^2} = \sqrt{5^2 - 4^2} = 3$. On the other hand, $FD = \frac{1}{2}BE = 2$. Therefore

$$DC = \sqrt{CF^2 - FD^2} = \sqrt{4^2 - 2^2} = 2\sqrt{3}.$$

It follows that $AD = DE = 2\sqrt{3} - 3$. Thus

$$\begin{aligned}[ABC] &= [ABE] + [BEC] \\ &= \frac{1}{2}(4)\left[2\left(2\sqrt{3}-3\right)\right] + \frac{1}{2}(4)(3) \\ &= 8\sqrt{3} - 6\end{aligned}$$

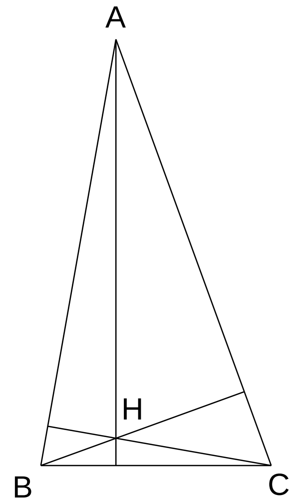


17. Note that H is the orthocentre of ΔABC. We have $\angle HBC = 20°$, $\angle HCB = 10°$, $\angle BAH = 80°$ and $\angle ABH = \angle ACH = 60°$. Applying sine law in ΔBHC, we have

$$\begin{aligned}\frac{2}{\sin 150°} &= \frac{BH}{\sin 10°} \\ BH &= 4\sin 10°\end{aligned}$$

Applying sine law in ΔABH, we have

$$\begin{aligned}\frac{4\sin 10°}{\sin 10°} &= \frac{AH}{\sin 60°} \\ AH &= 2\sqrt{3}\end{aligned}$$



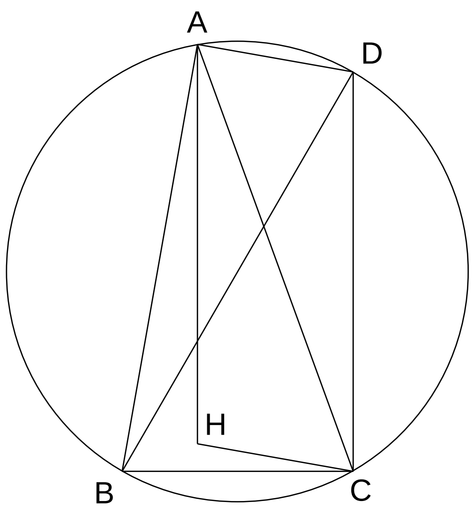
Alternative Solution

Note that H is the orthocentre of ΔABC. Construct the circumcircle of ΔABC and the diameter BD. The diameter of the circle is

$$\frac{BC}{\sin A} = \frac{2}{\sin 30°} = 4.$$

Since AH and DC are both perpendicular to BC, they are parallel. Similarly, AD and HC are parallel. So $AHCD$ is a parallelogram, and thus

$$\begin{aligned}AH = DC &= \sqrt{BD^2 - BC^2} \\ &= \sqrt{4^2 - 2^2} \\ &= 2\sqrt{3}\end{aligned}$$



18. We note that x and y cannot be too large. Starting with $A=\{1\}$ and applying the greedy algorithm, we try to append the smallest possible element into A each time. By this process get

$$A=\{1, 2, 3, 4, 5, 6, 8, 11, 18, 45\},$$

after which we can no longer append more elements into A. So A has at most 10 elements.

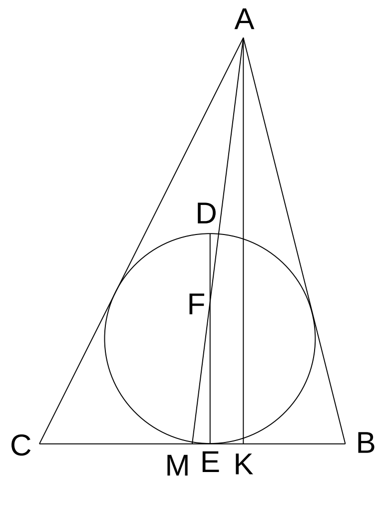
Let $A=\{a_1, a_2, \ldots, a_n\}$. We will show that $n \le 10$, so that the answer is indeed 10. Suppose $1 \le a_1 < a_2 < \cdots < a_n$. Then $a_{i+1}-a_i \ge \frac{a_{i+1}a_i}{30}$, or $\frac{1}{a_i}-\frac{1}{a_{i+1}} \ge \frac{1}{30}$ for $1 \le i \le n-1$. Summing from $i=5$ to $i=n-1$, we get $\frac{1}{a_5}-\frac{1}{a_n} \ge \frac{n-5}{30}$. So $\frac{1}{a_5} \ge \frac{n-5}{30}$. Noting that $a_i \ge i$ for all i, this implies $5 \le a_5 < \frac{30}{n-5}$, from which we get $n \le 10$ as desired.

19. Assume $m \ge n$. Then k is good if and only if $\log_{2003} m - \log_{2003} n < \log_{2003} k < \log_{2003} m + \log_{2003} n$. This is equivalent to $\frac{m}{n} < k < mn$. According to the question, there are 100 possible integral values of k. It is easy to see that n cannot be 1. We claim that $mn \le 134$. Otherwise, we have

$$mn-\frac{m}{n} \ge mn-\frac{mn}{n^2} \ge mn-\frac{mn}{4}=\frac{3mn}{4} \ge \frac{3(135)}{4} > 101$$

so that there would be more than 100 good numbers. Moreover, when $mn = 134$ with $m = 67$, $n = 2$, k can take one of the 100 values 34, 35, 36, …, 133. Hence the maximum value of mn is 134.

20. Let M be the mid-point of BC and K be the foot of the perpendicular from A to BC. Let also $AK = h$, s be the semi-perimeter and r be the radius of the inscribed circle. With standard notations, we have



$$CK = b\cos C = \frac{a^2+b^2-c^2}{2a}$$

$$CE = s-c$$

$$ME = CE - CM = \frac{b-c}{2}$$

$$MK = CK - CM = \frac{b^2-c^2}{2a}$$

Noting that $\Delta MEF \sim \Delta MKA$, we have $\frac{FE}{AK}=\frac{ME}{MK}=\frac{a}{b+c}=\frac{1}{6}$, i.e. $FE=\frac{1}{6}h$.

On the other hand, we have $\frac{ah}{2}=rs$ since both are equal to the area of ΔABC.

As a result, $DE = 2r = \frac{a}{s}h = \frac{2}{7}h$. So the answer is $\left(\frac{2}{7}-\frac{1}{6}\right):\frac{1}{6}=\frac{5}{7}$.