International Mathematical Olympiad
Hong Kong Preliminary Selection Contest 2020

Outline of solutions

Answers:

1.	25	**2.**	20	**3.**	240	**4.**	234567891
5.	19	**6.**	$\frac{32}{3}$	**7.**	$\frac{-1+\sqrt{29}}{2}$	**8.**	482020
9.	66.7	**10.**	4544	**11.**	5	**12.**	246
13.	5	**14.**	9	**15.**	285120	**16.**	$\frac{2}{7}$
17.	45	**18.**	181	**19.**	816	**20.**	515

Solutions:

1. We have

$$\begin{aligned} n &= (10^{2020}+2020)^2 \\ &= 10^{4040} + 2\cdot 2020\cdot 10^{2020} + 2020^2 \\ &= 1\,\underbrace{00...00}_{4040\text{ zeros}}+4040\,\underbrace{00...00}_{2020\text{ zeros}}+4080400 \\ &= 100...0040400...004080400 \end{aligned}$$

The sum of the digits of n is thus $1+4+4+4+8+4=25$.

2. We have $x(1+y+yz)=37$. As 37 is prime, we must have $x=1$ and $1+y+yz=37$. The latter implies $y(1+z)=36$. Given $x<y<z$, we should take $y=2$ and $1+z=18$ for $y+z$ to be maximised. Hence the greatest possible value of $x+y+z$ is $1+2+17=20$.

3. If there are n^2 pieces of bricks, n of them will be removed, leaving n^2-n pieces. Since $(n-1)^2<n^2-n<n^2$, another $n-1$ pieces will be removed in the next round, leaving $n^2-n-(n-1)=(n-1)^2$ pieces. Hence, starting with 2020^2, the number of pieces remaining after each removal will follow the sequence

$$2020^2 \to 2020\cdot 2019 \to 2019^2 \to 2019\cdot 2018 \to 2018^2 \to \cdots$$

Eventually it becomes

$$\cdots \to 17^2 \to 17\cdot 16 \to 16^2 \to 16\cdot 15.$$

As $16^2>250>16\cdot 15$, the answer is $16\cdot 15=240$.

4. Clearly I should be 1 (if another letter is 1, then one can easily check that swapping the value of this letter with I leads to a lower score). Note that for integers x, y, z greater than 1, we have

$$x < y \quad \text{implies} \quad x^{y^z} > y^{x^z}.$$

(This is intuitively obvious by trying a few small cases, and a technical proof is given in the remark below.) It follows that one should choose 234567891, for otherwise one can swap a digit with a smaller one on its right (except the ending 1) to get a higher score.

Remark. To show that $x^{y^z} > y^{x^z}$ whenever $x < y$, we consider two cases:

- Suppose $x = 2$. When $y \geq 5$, it follows from a simple induction (on y) that $x^y > y^x$ and so

$$x^{y^z} = (x^y)^{y^{z-1}} > (x^y)^{x^{z-1}} > (y^x)^{x^{z-1}} = y^{x^z}.$$

  If $y = 3$ or $y = 4$ we can also use induction (on z) to show $x^{y^z} > y^{x^z}$ for all integers $z > 1$.

- Suppose $x \geq 3$. We shall prove that we must have $x^y > y^x$, or equivalently, $x^{1/x} > y^{1/y}$, and so the same argument above (in case $x = 2$ and $y \geq 5$) would work. We present two different proofs here:

  (1) Using calculus — the function $f(x) = x^{1/x}$ is decreasing when $x \geq 3$ since $f'(x) = x^{\frac{1}{x}-2}(1 - \ln x) < 0$ whenever $x > e$.

  (2) Without calculus — using binomial theorem, we have, for integer $k \geq 3$,

$$\begin{aligned} k^{k+1} &= \underbrace{k^k + k^k + \cdots + k^k}_{k \text{ terms}} \\ &> \underbrace{k^k + k^k + \cdots + k^k}_{k-1 \text{ terms}} + k^2 + 1 \\ &> k^k + \binom{k}{1}k^{k-1} + \binom{k}{2}k^{k-2} + \cdots + \binom{k}{k-2}k^2 + k^2 + 1 \\ &= (k+1)^k \end{aligned}$$

  and so $k^{1/k} > (k+1)^{1/(k+1)}$. Recurring this inequality eventually gives $x^y > y^x$.

5. Note that there cannot be two consecutive boys. (To see this, take the longest chain of consecutive boys. Then the last boy is next to a boy and a girl, so this boy would be telling the truth.) Hence there must be some number (one or more) of girls between two boys, and so the students on the circle are of the form

$$(BG \cdots G)(BG \cdots G) \cdots (BG \cdots G)$$

where B and G denote a boy and a girl respectively. We call the students within each pair of parentheses a *block*, which consists of exactly one boy and one or more girls. A girl is lying (i.e. the two students next to her are of the same gender) if and only if

- she is the only girl in the block; or

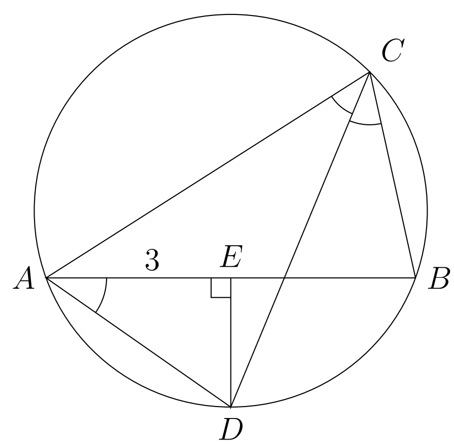
- she is in a block of three or more girls and she is neither the first nor the last girl.

Since $28 \equiv 1 \pmod 3$, one of the lying girls is of the former type and two are of the latter type. (If all are of the former type, then there are three (BG) and the rest are (BGG), so the number of students is divisible by 3. Similar contradictions arise if all students are of the latter type, or if two are of the former type and one is of the latter type.) It follows that there is exactly one (BG) block. The remaining 26 students thus form 8 blocks — each of them being (BGG) with two lying girls each to be inserted into one of the blocks (possibly the same one). Hence there are 9 blocks in total, meaning that there are 9 boys and hence $28 - 9 = 19$ girls in the class.

6. By the cosine formula, we have

$$6^2 = 7^2 + 8^2 - 2(7)(8)\cos C,$$

which gives $\cos C = \frac{11}{16}$. Let E be the mid-point of AB so that $AE = 3$. Then $DE \perp AB$ and CD bisects $\angle ACB$.



It follows that $\angle EAD = \angle BCD = \frac{C}{2}$ and so

$$AD^2 = \left(\frac{AE}{\cos \angle EAD}\right)^2 = \frac{9}{\cos^2 \frac{C}{2}} = \frac{9}{\frac{1}{2}(1+\cos C)} = \frac{9}{\frac{1}{2}(1+\frac{11}{16})} = \frac{32}{3}.$$

7. Squaring both sides of the equation, we get $7 - x = (7 - x^2)^2$, which is equivalent to

$$7^2 - (2x^2+1)(7) + (x^4 + x) = 0.$$

This shows that 7 is a root to the following quadratic equation in t:

$$t^2 - (2x^2+1)t + (x^4 + x) = 0.$$

By the quadratic formula, the above equation has solution

$$t = \frac{2x^2 + 1 \pm \sqrt{(2x^2+1)^2 - 4(x^4+x)}}{2} = x^2 + x \text{ or } x^2 - x + 1.$$

Hence either $x^2+x=7$ or $x^2-x+1=7$, which give positive roots $x=\frac{1}{2}(-1+\sqrt{29})$ and $x=3$ respectively. The latter is rejected as it will make the right hand side of the original equation negative. We then check that the former satisfies the original equation.

Remark. It may be easier to solve the problem by brute force — square both sides and transpose terms to get $x^4-14x^2+x+42=0$. Then test factors of 42 to find that -2 and 3 satisfy the equation, so we obtain the factorisation $(x+2)(x-3)(x^2+x-7)=0$ and can then proceed as before.

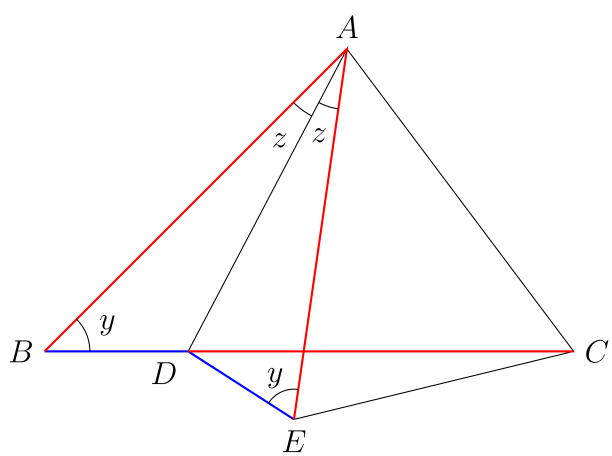
8. Let the number be $10000n+2020$. Since 77 divides 1001, we have $1000 \equiv -1 \pmod{77}$ and so we have

$$0 \equiv 10000n+2020 = 1000(10n+2)+20 \equiv (-1)(10n+2)+20 = 18-10n \pmod{77}.$$

Clearly we have $n>1$, so $10n-18$ is a positive multiple of 77 with unit digit 2. The smallest such multiple is $77 \times 6 = 462$, corresponding to $n=48$. The answer is thus 482020.

Remark. A more straightforward but also more tedious way of finding the answer is to let it be $77k$ and then investigate the last few digits of k one by one based on the information that $77k$ ends with 2020. For instance the unit digit of k must be 0, and then the tens digit of k must be 6, and the hundreds and thousands digits of k can be worked out in a similar manner.

9. Let $y=46.6^\circ$ and $z=20.1^\circ$. The key observation is that $3y+2z=180^\circ$, and we try to make constructions that allow this property to be used. To this end, we reflect $\triangle ABD$ across AD to get another congruent triangle $\triangle AED$.

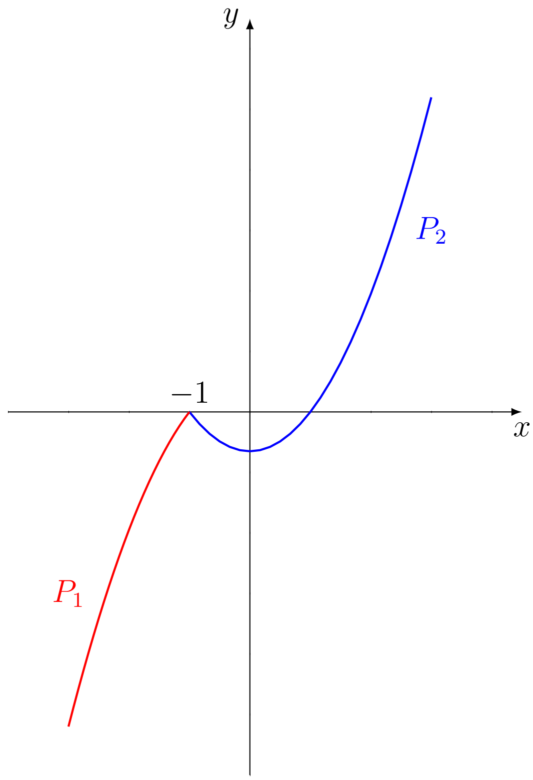


We have $\angle ADE = 180^\circ - y - z = 2y+z$ and $\angle ADC = y+z$. It follows that $\angle CDE = y$. As $AB = CD$, we see that $\triangle ABD$ is congruent to $\triangle CDE$. This means $\angle DCE = \angle BAD = z = \angle DAE$, so that A, D, E, C are concyclic. Hence

$$\angle CAD = 180^\circ - \angle DEC = 180^\circ - \angle BDA = y+z,$$

i.e. $x = 46.6+20.1 = 66.7$.

10. The graph of $y = (x-1)|x+1|$ is as follows:



The graph is essentially the parabola $P : y = x^2 - 1$, but with the part where $x < -1$ reflected across the x-axis. Call the reflected part (where $x < -1$) P_1 and the other part (where $x \geq 1$) P_2.

On the other hand, the graph of $y = x + \frac{k}{2020}$ is a straight line L with slope 1. Clearly L can intersect P_2 at most twice, while L can intersect P_1 at most once (this is intuitively clear from a rough sketch; a proof is given in the remark below). Hence, for the equation in the question to have three distinct real roots, L must intersect P_1 at one point and P_2 at two distinct points.

- L intersects P_1 at one point if and only if its y-intercept $\frac{k}{2020}$ is less than 1 (so that L would pass through $(-1, y)$ for some $y < 0$, i.e. we need $k < 2020$.
- L intersects P_2 at two distinct points if and only if the equation $(x+1)(x-1) = x + \frac{k}{2020}$ (which is equivalent to $2020x^2 - 2020x - (k + 2020) = 0$) has positive determinant, i.e. we need

  $$0 < 2020^2 + 4(2020)(k + 2020) = 2020(4k + 10100) = 8080(k + 2525),$$

  or $k > -2525$.

It follows that the possible values of k are -2524, -2523, ..., 2019, and so the answer is $2019 - (-2524) + 1 = 4544$.

Remark. To prove that the two graphs

$$L : y = x + \frac{k}{2000} \quad \text{and} \quad P_1 : y = 1 - x^2 \text{ (where } x < -1)$$

intersect at most once, one may use calculus — L has slope 1 while the derivative of P_1 is greater than 1 throughout (the derivative is $-2x$, which is greater than 2 whenever $x < -1$). For a non-calculus proof, one may also consider the equation

$$x + \frac{k}{2020} = 1 - x^2$$

and note that its sum of roots is -1, which means it cannot have two roots both being less than -1.

11. Setting $x = y + 2$, the equation becomes

$$(y+2)^3 - (k+1)(y+2)^2 + k(y+2) + 12 = 0,$$

or upon simplification,

$$y^3 + (5-k)y^2 + (8-3k)y + (16-2k) = 0.$$

Let $A = a - 2$, $B = b - 2$ and $C = c - 2$. Then $A^3 + B^3 + C^3 = -18$, and A, B, C are roots of the above equation. Hence we have

$$A + B + C = k - 5, \quad AB + BC + CA = 8 - 3k \quad \text{and} \quad ABC = 2k - 16.$$

Since $(A+B+C)^3 = (A^3+B^3+C^3) + 3(A+B+C)(AB+BC+CA) - 3ABC$, we have

$$(k-5)^3 = -18 + 3(k-5)(8-3k) - 3(2k-16),$$

which simplifies to $(k-5)(k^2-k+7) = 0$. As $k^2 - k + 7 = 0$ has no real root, we have $k = 5$.

12. Clearly $S \leq 129 \times 2 = 258$. But 258 does not work, as a simple counterexample is given by the case where 8 balls have 13 written and 11 balls have 14 written (note that $13 \times 8 + 14 \times 11 = 258$). In this case no matter how the balls are divided into 2 piles, one pile would contain at least 10 balls so that the sum of the numbers on the balls will be at least 130.

By changing the number on one ball from 14 to 13, we get a counterexample for 257. As we have 11 balls with 14 written, we also get counterexamples for 256, 255, ..., 247 when we change the number on a ball from 14 to 13 each time. (Note that $13 \times 19 = 247$, and the counterexample for 247 is with 19 balls all with 13 written.)

Now we show that $S = 246$ works. In that case we just keep adding balls to one pile until it is no longer possible to keep the sum at 129 or below. Then as we stop, the sum of the numbers on the balls added is at least 116 (if it is 115 or less we could at least add one more ball as the number on a ball is at most 14). But the sum cannot be exactly 116, for otherwise all remaining balls must have 14 written, yet $246 - 116$ is not divisible by 14. That means the sum is at least 117, so the sum of the numbers on the remaining balls is at most $246 - 117 = 129$ and we have successfully divided the balls into two piles subject to the requirement. It follows that the answer is 246.

13. We first note that the 5 numbers -1, 1, 3, 5, 7 satisfy the requirement. To show that n cannot be greater than 5, note that there can be at most one multiple of 3 on the blackboard, for both the sum of and difference between two multiples of 3 is divisible by 3 (and hence cannot be a power of 2). Hence, if n were to be greater than 5, there must be at least 5 non-multiples of 3 on the blackboard. We consider the following two cases, each of which leads to a contardiction:

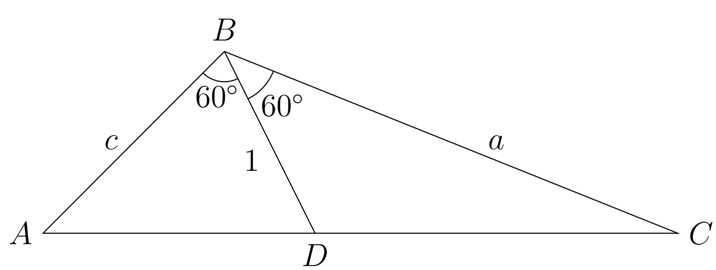
- Suppose four of them are congruent to 1 modulo 3 (same if four of them are congruent to 2 modulo 3), say, $a \equiv b \equiv c \equiv d \equiv 1 \pmod 3$. Then the difference between any two of a, b, c, d is divisible by 3 and cannot be a power of 2. Hence the sum of any two of them is a power of 2. In particular, that means $S_1 = a + b$, $S_2 = c + d$, $S_3 = a + c$ and $S_4 = b + d$ are all powers of 2. However, we have $S_1 + S_2 = S_3 + S_4$ with S_1 equal to neither S_3 nor S_4. This is impossible (by considering the binary representation of $S_1 + S_2$ and $S_3 + S_4$).
- Suppose three of them are congruent to 1 modulo 3 and two of them are congruent to 2 modulo 3 (same if it is the other way round), say, $a \equiv b \equiv c \equiv 1 \pmod 3$ and $d \equiv e \equiv 2 \pmod 3$. As before, each of $a + b$, $a + c$ and $b + c$ is a power of 2.

  Note that d cannot be larger than two of a, b, c, for if $d > a$ and $d > b$ then $d - a$ and $d - b$ are powers of 2 (as $d + a$ and $d + b$ are divisible by 3), so $S_1 = d - a$, $S_2 = a + c$, $S_3 = d - b$ and $S_4 = b + c$ are all powers of 2 with $S_1 + S_2 = S_3 + S_4$. As $S_1 \neq S_3$, we must have $S_1 = S_4$, which however is impossible since $S_1 \equiv 1 \pmod 3$ while $S_4 \equiv 2 \pmod 3$.

  Hence d is larger than at most one of a, b, c, and hence smaller than at least two of them, say, $d < a$ and $d < b$. Then $S_1 = b + c$, $S_2 = a - d$, $S_3 = a + c$ and $S_4 = b - d$ are all powers of 2 with $S_1 + S_2 = S_3 + S_4$. As $S_1 \neq S_3$, we must have $S_1 = S_4$, or $c = -d$. Hence c is negative and d is positive (for otherwise we also have $d < c$ so replacing b by c in the above argument would give $b = -d$ which contradicts $b \neq c$), and hence a and b are positive. But then replacing d by e in the above argument would give $c = -e$ which contradicts $d \neq e$.

We thus conclude that there cannot be 5 non-multiples of 3 on the blackboard, and so the greatest possible value of n is 5.

14. Let $AB = c$ and $BC = a$.



As $[ABC] = [ABD] + [CBD]$ (where $[XYZ]$ denotes the area of XYZ), we have

$$\frac{1}{2}ca\sin 120^\circ = \frac{1}{2}c(1)\sin 60^\circ + \frac{1}{2}a(1)\sin 60^\circ,$$

which simplifies to $ac = a + c$. Hence

$$4BC + AB = 4a + c = \frac{(4a+c)(a+c)}{ac} = \frac{4a^2 + 5ac + c^2}{ac} = 5 + \frac{4a}{c} + \frac{c}{a}.$$

Note that

$$\frac{4a}{c} + \frac{c}{a} = \left(\sqrt{\frac{4a}{c}} - \sqrt{\frac{c}{a}}\right)^2 + 4$$

which is at least 4. Hence the minimum value of $4BC + AB$ is 9, which is attained when $\frac{4a}{c} = \frac{c}{a}$ (together with $ac = a + c$ one can solve to get $a = \frac{3}{2}$ and $c = 3$).

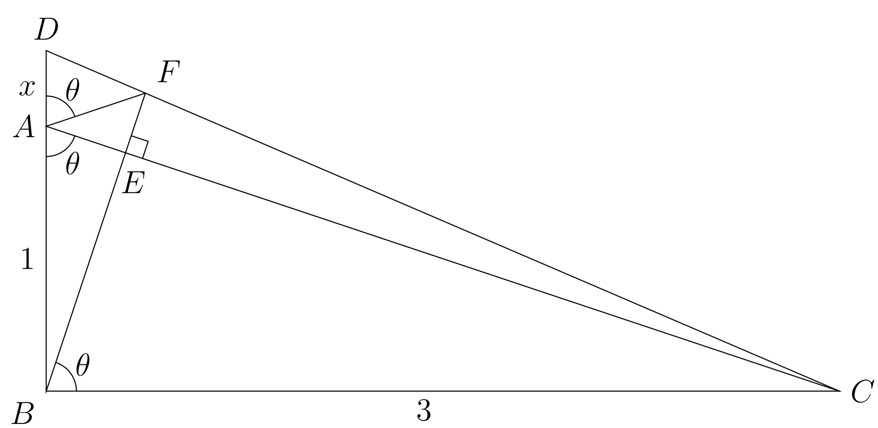
15. Suppose we permute the digits 0 to 9 to form a ten-digit number ABCDEFGHIJ. Note that the sum of digits is $0 + 1 + 2 + \cdots + 9 = 45$ which is divisible by 9, so the number formed must be divisible by 9, and it remains to ensure that the number is also divisible by 11 and does not start with 0.

    Let the digits A, C, E, G, I form one group and the rest form another. The number is divisible by 11 if and only if the sum of one group is congruent to the sum of the other group modulo 11. As the sum of these two sums is $45 \equiv 1 \pmod{11}$, the sum of each group should be congruent to 6 modulo 11. Furthermore the sum of each group is at least $0 + 1 + 2 + 3 + 4 = 10$. Hence the two sums must be 17 and 28. There are 11 ways to pick five of 0 to 9 with sum 17 (the complement of each has sum 28):

    $$\begin{array}{cccc} \{0,1,2,5,9\} & \{0,1,2,6,8\} & \{0,1,3,4,9\} & \{0,1,3,5,8\} \\ \{0,1,3,6,7\} & \{0,1,4,5,7\} & \{0,2,3,4,8\} & \{0,2,3,5,7\} \\ \{0,2,4,5,6\} & \{1,2,3,4,7\} & \{1,2,3,5,6\} & \end{array}$$

    For each of these 11 ways, there are 9 possible positions to place the digit 0 (as it cannot be the leftmost digit), and then 4! ways to place the digits in the same group of 0, and finally 5! ways to place the digits in the other group. The answer is thus $11 \times 9 \times 4! \times 5! = 285120$.

16. Let $AB = x$ and $\angle DAF = \angle BAC = \theta$. Note that $\angle CBE = \theta$ as well.



Note that $\dfrac{[BDF]}{[BCF]} = \dfrac{DF}{CF} = \dfrac{[ADF]}{[ACF]}$, where $[XYZ]$ denotes the area of XYZ. This gives

$$\frac{\frac{1}{2}BD \cdot BF \sin(90^\circ - \theta)}{\frac{1}{2}BC \cdot BF \sin\theta} = \frac{\frac{1}{2}AD \cdot AF \sin\theta}{\frac{1}{2}AC \cdot AF \sin(180^\circ - 2\theta)},$$

which simplifies to $\dfrac{BD\cos\theta}{BC\sin\theta} = \dfrac{AD\sin\theta}{AC\sin 2\theta}$, or $2AC \cdot BD\cos^2\theta = AD \cdot BC\sin\theta$. Using

$$\sin\theta = \frac{3}{\sqrt{10}} \quad \text{and} \quad \cos\theta = \frac{1}{\sqrt{10}},$$

we get

$$2\sqrt{10}(1+x)\left(\frac{1}{10}\right) = 3x \cdot \frac{3}{\sqrt{10}}$$

which gives $x = \dfrac{2}{7}$.

17. Let $f(x,y) = \sqrt{(x+2020)(y+2020)} - \sqrt{2020}(\sqrt{x} + \sqrt{y})$. Adding the four equations in the system gives

$$f(a,b) + f(b,c) + f(c,d) + f(d,a) = 0.$$

Note that $f(x,y) \geq 0$ for any x, y since

$$\begin{aligned}(x+2020)(y+2020) &= xy + 2020^2 + 2020(x+y) \\ &= (\sqrt{xy} - 2020)^2 + 4040\sqrt{xy} + 2020(x+y) \\ &\geq 4040\sqrt{xy} + 2020(x+y) \\ &= \left[\sqrt{2020}\left(\sqrt{x} + \sqrt{y}\right)\right]^2\end{aligned}$$

It follows that we must have $f(a,b) = f(b,c) = f(c,d) = f(d,a) = 0$. From the proof of $f(x,y) \geq 0$ we see that $f(x,y) = 0$ if and only if $xy = 2020^2$. Hence we now need $ab = bc = cd = da = 2020^2$, and we check that any quadruple (a,b,c,d) satisfying this would satisfy $a = c$ and $b = d$, and hence the original system of equations. We only need to choose a and its value automatically fixes b, c and d. As $2020^2 = 2^4 \cdot 5^2 \cdot 101^2$, it has $(4+1)(2+1)(2+1) = 45$ positive factors. Hence there are 45 choices of a, each of which leads to a quadruple (a,b,c,d) of positive integers that satisfies the system of equations in the question. The answer is thus 45.

18. As the sum of the interior angles of an 11-sided polygon is $180^\circ \times 9$, there can be at most 8 reflex angles in an 11-sided polygon.

To form an 11-sided polygon with 8 reflex angles, we choose 8 of the 11 interior angles to be reflex, and there are $\binom{11}{8}$ ways to do so. However, some of the choices are the same upon rotation. Indeed, the same type of polygon is counted 11 times as one can rotate a certain type of 11-sided polygon to obtain 11 different configurations (note that since 11 is prime the 11 configurations must all be different as long as the interior angles are not all reflex or not all non-reflex). Hence the number of types of 11-sided polygons with 8 reflex angles is

$$\frac{\binom{11}{8}}{11}$$

and similarly this number is $\dfrac{\binom{11}{n}}{11}$ if the number of reflex angles is $n > 0$. Summing n from 1 to 8, and not forgetting the one type of convex 11-sided polygons (corresponding to $n = 0$), we get the answer

$$1 + \frac{\binom{11}{1} + \binom{11}{2} + \cdots + \binom{11}{8}}{11} = 1 + \frac{2^{11} - \binom{11}{0} - \binom{11}{9} - \binom{11}{10} - \binom{11}{11}}{11} = 181.$$

It is also not hard to see that every such type of 11-sided polygons does exist.

19. There are 4! ways to arrange the order of the women from left to right. Assume a certain order of the women and we count the number of seating arrangements. Note that if two women are not adjacent there must be at least two men between them, and that if a woman is not adjacent to any other woman she must sit at either end. Hence there are four cases, depending on how the four women are separated:

- If the women are split $1 + 2 + 1$, then the arrangement must be $W\square\square WW\square\square W$ where W denotes a woman and $\square$ denotes a man. With the order of the women fixed, each man must sit next to his wife and so there is only 1 seating arrangement in this case.
- If the women are split $2 + 2$, there are three subcases:
  - If both ends are occupied by men, the arrangement must be $\square WW\square\square WW\square$ and each man must sit next to his wife.
  - If the two ends are occupied by a man and a woman, there are 2 ways to choose an end for woman, say, left, then we have $WW\square\square ???\square$ and 2 further ways to put the other two women in the three ?'s. Again each such arrangement fixes all the men.
  - If both ends are occupied by women, the arrangement must be $WW\square\square\square\square WW$. This time only two men are fixed and there are 2! ways to permute the 2 men in the middle.

  Hence there are altogether $1 + 2 \times 2 + 2! = 7$ seating arrangement in this case.
- If the women are split $1 + 3$ or $3 + 1$, there are 2 ways to place the singleton woman at either end (say, the left) and then two subcases:
  - If the other end is also occupied by a woman, the arrangement must be $W\square\square\square\square WWW$ and there are 2! ways to permute the 2 men in the middle.
  - If the other end is occupied by a man, the arrangement is $W\square\square ????\square$ and there are 2 ways to put the three women in the four ?'s.

  Hence there are altogether $2 \times (2! + 2) = 8$ seating arrangement in this case.
- If all four women are adjacent, there are two subcases:

- If one end is occupied by a woman, there are 2 ways to choose the end, say left, and then the arrangement must be $WWWW\square\square\square\square$. Only one man is fixed and there are 3! ways to permute the rest.
- If both ends are occupied by men, the arrangement is $\square??????\square$ and there are 3 ways to put the four women in the six ?'s. Each of them fixes two men and there are 2! ways to permute the remaining men.

Hence there are altogether $2 \times 3! + 3 \times 2! = 18$ seating arrangement in this case.

It follows that the answer is $4! \times (1 + 7 + 8 + 18) = 816$.

20. Let F_n denote the n-th term of the sequence. We need to find the remainder when F_{2020} is divided by 1000. It suffices to find the remainder when F_{2020} is divided by 8 and 125 respectively.

- Taking modulo 8 on the sequence, we get

$$1, 1, 2, 3, 5, 0, 5, 5, 2, 7, 1, 0, 1, 1, ...$$

and we see that it repeats every 12 terms. Since $2020 \equiv 4 \pmod{12}$, we have $F_{2020} \equiv F_4 \equiv 3 \pmod 8$.

- To find the remainder when F_{2020} is divided by 125, we have

$$\begin{aligned}
F_{2020} &= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{2020} - \left(\frac{1-\sqrt{5}}{2}\right)^{2020}\right] \\
&= \frac{1}{2^{2020}\sqrt{5}}\left[\left(1+\sqrt{5}\right)^{2020} - \left(1-\sqrt{5}\right)^{2020}\right] \\
&= \frac{1}{2^{2020}\sqrt{5}} \cdot 2 \cdot \left[\binom{2020}{1}\sqrt{5} + \binom{2020}{3}\left(\sqrt{5}\right)^3 + \binom{2020}{5}\left(\sqrt{5}\right)^5 + \cdots\right] \\
&= \frac{1}{2^{2019}}\left[\binom{2020}{1} + \binom{2020}{3} \cdot 5 + \binom{2020}{5} \cdot 5^2 + \cdots\right]
\end{aligned}$$

Note that F_{2020} is an integer, and the '$\cdots$' part consists of 5^3 and higher powers of 5 (hence divisible by 125). It follows that

$$2^{2019}F_{2020} \equiv \binom{2020}{1} + 5\binom{2020}{3} + 25\binom{2020}{5} \pmod{125}.$$

Direction computation (details in the remark below) yields $102 \cdot 2^{2019} \equiv 1 \pmod{125}$, $5\binom{2020}{3} \equiv 75 \pmod{125}$ and $25\binom{2020}{5} \equiv 100 \pmod{125}$. Hence

$$\begin{aligned}
F_{2020} &= 102 \cdot 2^{2019}F_{2020} \\
&= 102\left[\binom{2020}{1} + 5\binom{2020}{3} + 25\binom{2020}{5}\right] \\
&\equiv 102(20 + 75 + 100) \\
&\equiv 15 \pmod{125}
\end{aligned}$$

As $F_{2020} \equiv 15 \pmod{125}$, the last three digits of F_{2020} may be 015, 140, 265, 390, 515, 640, 765 or 890. Among these, only 515 is congruent to 3 modulo 8, and hence is the answer to the question.

Remark.

- To find n for which $2^{2019}n \equiv 1 \pmod{125}$, we note that $2^7 \equiv 3$, $3^5 \equiv -7$ and $7^4 \equiv 26 \pmod{125}$. Hence we have

$$\begin{aligned}
2^{2019} &= 2^3(2^7)^{288} \\
&\equiv 8(3)^{288} = 8 \cdot 3^3(3^5)^{57} \\
&\equiv (-34)(-7)^{57} = 34 \cdot 7(7^4)^{14} \\
&\equiv (-12)(26)^{14} = -12 \cdot 676^7 \\
&\equiv (-12)(51)^7 = (-12)(51)(2601)^3 \\
&\equiv (-12)(51)(-24)^3 = 2^3 12^4 51 \\
&\equiv 38 \pmod{125}
\end{aligned}$$

  (The above could be simplified if one knows Euler's theorem, which asserts that $2^{100} \equiv 1 \pmod{125}$ since $\phi(125) = 100$. In that case one immediately gets $2^{2019} \equiv 2^{19} = 524288 \equiv 38 \pmod{125}$.) The problem is this reduced to $38n \equiv 1 \pmod{125}$. One can use the Euclidean algorithm to find that the greatest common factor of 38 and 125 to be 1, and tracing the divisions gives $125(7) - 38(23) = 1$ (or one may simply check the multiples of 125 modulo 38 and find that $125 \cdot 7 \equiv 1 \pmod{38}$). It follows that $n \equiv -23 \equiv 102 \pmod{125}$.

- To find the remainder when $5\binom{2020}{3}$ is divided by 125, note that

$$5\binom{2020}{3} = 5 \cdot \frac{2020 \cdot 2019 \cdot 2018}{6} = 25 \cdot \frac{404 \cdot 2019 \cdot 2018}{6} = 25 \cdot 202 \cdot 673 \cdot 2018,$$

  and as $202 \cdot 673 \cdot 2018$ has unit digit 8, it is of the form $5k + 3$ and so $5\binom{2020}{3} = 125k + 75 \equiv 75 \pmod{125}$.

- We can get $25\binom{2020}{5} \equiv 100 \pmod{125}$ in a similar manner.