香港資優教育學苑
The Hong Kong Academy for Gifted Education

International Mathematical Olympiad Preliminary Selection Contest – Hong Kong 2021

國際數學奧林匹克 — 香港選拔賽初賽 2021

29 May 2021 (Saturday)
2021 年 5 月 29 日（星期六）

Question Book
問題簿

Instructions to Contestants:
考生須知：

1. The contest comprises a 3-hour written test.
   比賽以筆試形式進行，限時三小時。

2. Questions are bilingual. Answer all questions.
   題目中英對照。全卷題目均須作答。

3. Put your answers on the answer sheet.
   請將答案寫在答題紙上。

4. The use of calculators is NOT allowed.
   不可使用計算機。

5. Measuring instruments like rulers, compasses, etc. can be used.
   直尺、圓規及其它量度工具可作輔助之用。

Co-organised by The Hong Kong Academy for Gifted Education,
the Gifted Education Section of the Education Bureau and
International Mathematical Olympiad Hong Kong Committee
香港資優教育學苑、教育局資優教育組及國際數學奧林匹克香港委員會合辦

1. 100 students, numbered 1 to 100, are divided into 20 groups, each with 5 students. The student with the largest number in each group becomes the group leader. What is the maximum number of group leaders whose number is prime? (1 mark)

現有 100 名學生，他們的學號分別為 1 至 100。他們被分成 20 組，每組 5 人，當中學號最大的學生成為組長。那麼，學號是質數的組長最多有幾人？ （1 分）

2. Let n be a three-digit number which is divisible by the product of its digits. Find the greatest possible value of n. (1 mark)

設 n 為三位數，使得 n 可被自己的數字之積整除。求 n 的最大可能值。 （1 分）

3. Let a and b be non-zero real numbers such that $\frac{a}{b}+\frac{b}{a}=5$ and $\frac{a^2}{b}+\frac{b^2}{a}=12$. Find the value of $\frac{1}{a}+\frac{1}{b}$. (1 mark)

設 a、b 為滿足 $\frac{a}{b}+\frac{b}{a}=5$ 和 $\frac{a^2}{b}+\frac{b^2}{a}=12$ 的非零實數。求 $\frac{1}{a}+\frac{1}{b}$ 的值。 （1 分）

4. Three students each randomly chooses a prime number less than 100, and then the teacher multiplies the three chosen primes together. What is the probability that the product obtained by the teacher is divisible by 2021? (1 mark)

三名學生各隨機選出一個小於 100 的質數，之後老師把這三個質數乘起來。老師所得的乘積可被 2021 整除的概率是多少？ （1 分）

5. Ann, Ben and Cat join a game as a team. Each of them has to independently answer 2021 yes-no questions (the questions are the same for everyone), in each of which either 'yes' or 'no' must be chosen. After that, the answers of Ann and Ben are compared and the number of questions in which they chose the same answer is recorded. The same is done for Ann's and Cat's answers, as well as for Ben's and Cat's answers. The score of the team in the game is the largest number recorded. Find the minimum possible score of the team in the game. (1 mark)

甲、乙、丙三人組隊參加一個遊戲。他們各自回答 2021 道是非題（每人的問題均相同），當中每題均須選擇「是」或「否」。之後比較甲、乙二人的答案並記錄他們選了相同答案的題數，再對甲、丙二人作同樣記錄，亦對乙、丙二人作同樣記錄。記錄中最大的一個數即為隊伍在遊戲中的得分。求他們在遊戲中得分的最小可能值。 （1 分）

6. Let G be the centroid of ΔABC. If the distance from G to the sides AB, BC and CA are 20, 12 and 15 respectively, find the length of BG. (1 mark)

設 G 為 ΔABC 的重心。若 G 到邊 AB、BC 和 CA 的距離分別是 20、12 和 15，求 BG 的長度。 （1 分）

7. Find the sum of all real roots of the equation $(2\sqrt[5]{x+1}-1)^4+(2\sqrt[5]{x+1}-3)^4=16$. (1 mark)

求方程 $(2\sqrt[5]{x+1}-1)^4+(2\sqrt[5]{x+1}-3)^4=16$ 的所有實根之和。（1 分）

8. Let n be a positive integer not exceeding 2021. If each of n, $n+1$ and $n+2$ can be expressed as the sum of two (possibly the same) non-negative integral powers of 2, find the sum of all possible values of n. (1 mark)

設 n 為不超過 2021 的正整數。若 n、$n+1$ 和 $n+2$ 均可表示成兩個（可以相同）2 的非負整數次方之和，求 n 的所有可能值之和。（1 分）

9. In a bag there are 2021 balls, numbered 1 to 2021. Two balls are drawn from the bag at random. What is the probability that the ratio of the numbers of the two balls drawn is an integral power of 2? (1 mark)

某袋子中有 2021 個分別編號為 1 至 2021 的球。現從袋子中隨機抽出兩個球，則兩球編號之比為 2 的某整數次方的概率是多少？（1 分）

10. A teacher wants to set some questions for a test on addition. Each question must involve adding a pair of two-digit positive integers (possibly the same), and no carry may be involved in the process. The answer must also be a two-digit number. How many different questions can the teacher set? (The result of swapping the order of two different numbers is regarded to be a different question, e.g. $12+34$ and $34+12$ are regarded as two questions.) (1 mark)

老師準備為一次加法測驗擬題。每題必定是把兩個兩位正整數（可以相同）相加，當中不能牽涉進位，答案亦必須是兩位數。老師共可擬定多少道不同的問題？（把兩個不同整數的次序互換視為不同題目，例如 $12+34$ 和 $34+12$ 視為兩道題目。）（1 分）

11. Suppose $(21x+20y)^{2021}+x^{2021}+484x+440y=0$ for some non-zero real numbers x and y. Find $\frac{x}{y}$. (2 marks)

設 x、y 為滿足 $(21x+20y)^{2021}+x^{2021}+484x+440y=0$ 的非零實數。求 $\frac{x}{y}$。（2 分）

12. *OABC* is a trapezium with *OC* // *AB* and $\angle AOB = 37°$. Furthermore, *A*, *B*, *C* all lie on the circumference of a circle centred at *O*. The perpendicular bisector of *OC* meets *AC* at *D*. If $\angle ABD = x°$, find *x*. (2 marks)

OABC 是梯形，其中 *OC* // *AB*，且 $\angle AOB = 37°$。此外，*A*、*B*、*C* 均位於一個以 *O* 為圓心的圓的圓周上。*OC* 的垂直平分線與 *AC* 交於 *D*。若 $\angle ABD = x°$，求 *x*。（2 分）

13. Find the smallest positive integer n such that the last four digits of n^3 (from left to right) are 2, 0, 2 and 1. (2 marks)

求最小的正整數 n，使得 n^3 的最後四位數字（從左至右）為 2、0、2 和 1。（2 分）

14. There are 21 students in a class, numbered 1 to 21. In each lesson, three students are responsible for preparing notes, and among them the one whose number is in the middle will give a presentation to the class. At the end of the academic year, it was found that for any three students, they have been preparing notes together exactly once. What will be the sum if we add up the number of the student giving presentation in each lesson? (2 marks)

某班有 21 名學生，他們的學號分別為 1 至 21。每堂課均由其中三名學生負責準備筆記，然後在三人當中學號排在中間的一人負責向全班作報告。學年過去後，發現班中任何三名學生均曾有剛好一次合作準備筆記。若我們把每堂課中負責報告的學生的學號加起來，所得的和是多少？（2 分）

15. There are some proper fractions with the following properties: the numerator and denominator are two-digit positive integers with exactly one digit in common, and this common digit is non-zero. When this common digit is removed the result is equal to the original fraction in lowest term (an example is $\frac{16}{64} \to \frac{1\not{6}}{\not{6}4} \to \frac{1}{4}$). Find the sum of all such fractions. (2 marks)

有些真分數滿足以下條件：分子和分母都是兩位正整數，當中有剛好一個共同的數字，而這個共同數字不等於零。當這個共同數字被刪去後，所得的結果剛好是原分數約至最簡後的形式（例如 $\frac{16}{64} \to \frac{1\not{6}}{\not{6}4} \to \frac{1}{4}$）。求所有這樣的分數之和。（2 分）

16. How many of the first 2021 positive integers can be expressed in the form $a^3+b^3+c^3-3abc$ where a, b, c are non-negative integers? (2 marks)

在首 2021 個正整數中，有多少個可寫成 $a^3+b^3+c^3-3abc$ 的形式，其中 a、b、c 為非負整數？（2 分）

17*. In ΔABC, $\angle BAC = 60°$ and M is a point on BC. Γ is the inscribed circle of ΔABC which touches sides BC, CA and AB at D, E and F respectively. AM meets Γ at P and Q with $AP < AQ$. If $AE = 4$, $EC = 2$ and $AP = QM$, find the length of AM. (2 marks)

在 ΔABC 中，$\angle BAC = 60°$，且 M 為 BC 上的一點。Γ 為 ΔABC 的內切圓，與邊 BC、CA 和 AB 分別相切於 D、E 和 F。AM 與 Γ 相交於 P 和 Q，其中 $AP < AQ$。若 $AE = 4$、$EC = 2$ 及 $AP = QM$，求 AM 的長度。（2 分）

*This question was cancelled in the contest due to a misprint in the original question paper.
由於原試卷中出現誤植，本題在比賽中被刪去。

18. Ann and Ben play a game as follows. Ann starts by constructing an acute-angled $\Delta A_0B_0C_0$ which is not equilateral and in which every interior angle (when measured in degrees) is an integer. For each $\Delta A_nB_nC_n$ constructed by one player, the other player subsequently constructs $\Delta A_{n+1}B_{n+1}C_{n+1}$ such that A_{n+1} is the foot of perpendicular from A_n to B_nC_n, and with B_{n+1} and C_{n+1} defined analogously. The game comes to an end when a $\Delta A_kB_kC_k$ constructed is not acute, and the player who constructs such a triangle can get k points while the other player gets $-k$ points. Assume that Ann is clever so she can construct $\Delta A_0B_0C_0$ in a way that maximises her score in the game. Find the greatest possible value of $\angle A_0$. (2 marks)

甲、乙二人進行以下遊戲：甲先構作一個非等邊的銳角 $\Delta A_0B_0C_0$，其中每隻內角（以「度」為單位時）均為整數。此後，每當某人構作 $\Delta A_nB_nC_n$ 後，另一人隨即構作 $\Delta A_{n+1}B_{n+1}C_{n+1}$，使得 A_{n+1} 為 A_n 到 B_nC_n 的垂足，而 B_{n+1} 和 C_{n+1} 亦以同樣方法定義。當某人構作出一個非銳角 $\Delta A_kB_kC_k$ 後遊戲便結束，而構作此三角形的人可得 k 分，另一人則得 $-k$ 分。假設甲是聰明的，從而她構作 $\Delta A_0B_0C_0$ 時會確保自己可以從遊戲中取得最高的分數。求 $\angle A_0$ 的最大可能值。（2 分）

19. For positive integer n, let $f(n)$ denote the number of ways of expressing n as the sum of the squares of two non-negative integers (without considering the order of the two numbers). For example, we have $f(25)=2$ since there are only two ways of writing 25 as the sum of two squares, namely, 0^2+5^2 and 3^2+4^2. Find the value of $f(3)+f(6)+f(9)+\cdots+f(2022)$. (2 marks)

對於正整數 n，設 $f(n)$ 表示把 n 寫成兩個非負整數平方和的方法數目（不考慮兩數的次序），例如 $f(25)=2$，因為只有 0^2+5^2 和 3^2+4^2 兩種方法把 25 寫成兩數的平方和。求 $f(3)+f(6)+f(9)+\cdots+f(2022)$ 的值。（2 分）

20. In ΔABC, $AB=15$ and $AC=13$. $DEFG$ is a square such that D is on AC, F is on AB and G is the mid-point of BC. If $AD=11$ and $AF>FB$, find the area of $DEFG$. (2 marks)

在 ΔABC 中，$AB=15$ 而 $AC=13$。$DEFG$ 是正方形，其中 D 位於 AC 上，F 位於 AB 上，且 G 是 BC 的中點。若 $AD=11$ 且 $AF>FB$，求 $DEFG$ 的面積。（2 分）

End of Paper

全卷完