International Mathematical Olympiad

Preliminary Selection Contest 2014 — Hong Kong

Outline of Solutions

Answers:

1. 1728	**2.** $\dfrac{45}{4}$	**3.** 5	**4.** 2029105
5. 135°	**6.** $2\sqrt{249}$	**7.** 110	**8.** 10
9. 13°	**10.** $\sqrt{26}$	**11.** 46	**12.** 15
13. $\dfrac{507}{10}$	**14.** 90	**15.** 1352737	**16.** 1432
17. 125	**18.** 546	**19.** $\dfrac{3}{2}+\sqrt{3}$	**20.** 3721

Solutions:

1. We have
$$\begin{aligned} x^3-y^3-36xy &= (x-y)(x^2+xy+y^2)-36xy \\ &= 12(x^2+xy+y^2)-36xy \\ &= 12(x^2+xy+y^2-3xy) \\ &= 12(x-y)^2 \\ &= 12^3 \\ &= 1728 \end{aligned}$$

   Remark. One may guess the answer by substituting suitable values of x and y.

2. It suffices to minimise $2|x+1|+4|x-5|+4\left|x-\dfrac{7}{2}\right|+|x-11|$ (which is 4 times the expression in the question), i.e. to find the minimum total distance from x to the 11 numbers: $-1, -1, \dfrac{7}{2}, \dfrac{7}{2}, \dfrac{7}{2}, \dfrac{7}{2}, 5, 5, 5, 5, 11$.

   By the triangle inequality, we have $|x-a|+|x-b| \ge |(x-a)+(b-x)| = b-a$ for any $a \le b$, and equality holds whenever x lies between a and b. From this, we try to pair up the numbers as follows: $(-1, 11), (-1, 5), \left(\frac{7}{2}, 5\right), \left(\frac{7}{2}, 5\right), \left(\frac{7}{2}, 5\right)$, leaving a single number $\frac{7}{2}$. From the above discussion, the total distance from x to each pair is minimised when x lies between them. Hence we can achieve the minimum by minimising the distance from x to $\frac{7}{2}$ while making sure that x lies between each pair of numbers. Clearly, this can be achieved by taking $x = \frac{7}{2}$. It follows that the answer is $\left|\frac{7}{2}+1\right| + 2\left|\frac{7}{2}-5\right| + \left|2\times\frac{7}{2}-7\right| + \left|\frac{\frac{7}{2}-11}{2}\right| = \frac{45}{4}$.

Remark. The total distance from x to a given set of numbers is minimised when x is the median of the set of numbers.

3. Note that $x^2 + y^2 - 4x - 4y + 8 = (x-2)^2 + (y-2)^2$ and $x^2 - 8x + 17 = (x-4)^2 + 1$. Hence if we let $A = (0, 0)$, $B = (2, y)$, $C = (x, 2)$ and $D = (4, 3)$, the expression in the question is equal to $AB + BC + CD$. This 'total length' is minimised when A, B, C, D are collinear (which is clearly possible: just take B to be the intersection of the line segment AD and the line $x = 2$, and likewise for C), and the minimum 'total length' is the length of AD, which is $\sqrt{(4-0)^2 + (3-0)^2} = 5$.

4. From $0 = f(f(0)) = f(b) = ab + b = (a+1)b$, we get $a = -1$ or $b = 0$.

   If $b = 0$, i.e. $f(x) = ax$, then we have $9 = f(f(f(4))) = f(f(4a)) = f(4a^2) = 4a^3$, which has no solution as a is an integer.

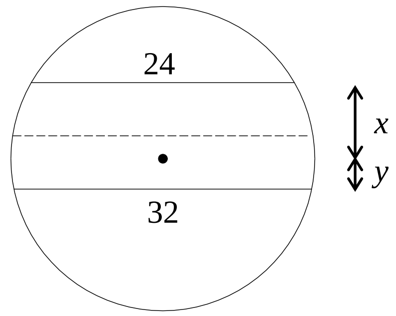
   Hence we must have $a = -1$. Then $f(x) = -x + b$ and hence $f(f(x)) = -(-x+b) + b = x$. It follows that $f(f(f(f(x)))) = f(f(x)) = x$ for all x and thus the answer is

$$1 + 2 + \cdots + 2014 = \frac{2014 \times 2015}{2} = 2029105.$$

5. Since sin P, sin Q and sin R are all positive, so are cos A, cos B and cos C. In other words, ΔABC is acute-angled. It follows that we must have $P = 90° \pm A$, $Q = 90° \pm B$ and $R = 90° \pm C$. Since PQR is a triangle, at most one of P, Q, R can be obtuse. If all of P, Q, R are acute, then $180° = P + Q + R = (90° - A) + (90° - B) + (90° - C) = 270° - 180° = 90°$, which is a contradiction. We conclude that exactly one of P, Q, R is obtuse. So without loss of generality we assume that $P = 90° + A$, $Q = 90° - B$ and $R = 90° - C$. In this case, we must have $180° = P + Q + R = (90° + A) + (90° - B) + (90° - C) = 270° + A - (180° - A) = 90° + 2A$. It follows that $A = 45°$ and so the largest angle is $P = 135°$.

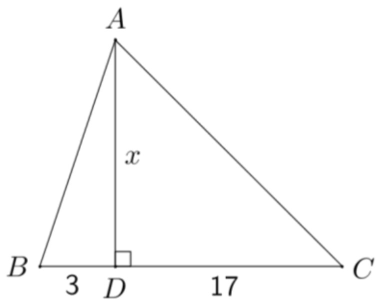
6. Let r be the radius of the circle. Denote by x and y the distances from the two chords to the centre of the circle. Since the perpendicular from the centre to a chord bisects the chord, by Pythagoras' Theorem we see that $\left(\frac{24}{2}\right)^2 + x^2 = r^2$, i.e. $12^2 + x^2 = r^2$. Similarly, we also have $16^2 + y^2 = r^2$. Combining the two equations gives $12^2 + x^2 = 16^2 + y^2$, which upon rearranging becomes $(x-y)(x+y) = x^2 - y^2 = 112$. There are two possibilities.

- If the two chords lie on the same half of the circle, then we have $x - y = 14$, which gives $x + y = 8$ and $y = -3$. This is clearly impossible.
- If the two chords lie on opposite halves of the circle, then we have $x + y = 14$, which gives $x - y = 8$. Solving the equations, we get $x = 11$ and $y = 3$. Hence $r = \sqrt{265}$. It is now clear that the desired chord is at a distance of $\frac{11-3}{2} = 4$ from the centre. Using Pythagoras' Theorem again, we find that its length is $2 \times \sqrt{\sqrt{265}^2 - 4^2} = 2\sqrt{249}$.



7. Let D be the foot of perpendicular from A to BC and let $AD = x$. Using the compound angle formula, we have

$$\begin{aligned}\frac{22}{7} &= \tan\angle CAB \\ &= \frac{\tan\angle BAD + \tan\angle CAD}{1 - \tan\angle BAD \tan\angle CAD} \\ &= \frac{\frac{3}{x} + \frac{17}{x}}{1 - \frac{3}{x} \times \frac{17}{x}} = \frac{20x}{x^2 - 51}\end{aligned}$$



After simplification, we get $11x^2 - 70x - 561 = 0$, i.e. $(x-11)(11x+51) = 0$. The only positive solution is $x = 11$. Thus the area of ΔABC is $\frac{(3+17)\times 11}{2} = 110$.

8. Let R, Y, G denote red, yellow and green ball respectively. Call a group 'pure' if all balls are of the same colour. We have the following cases:

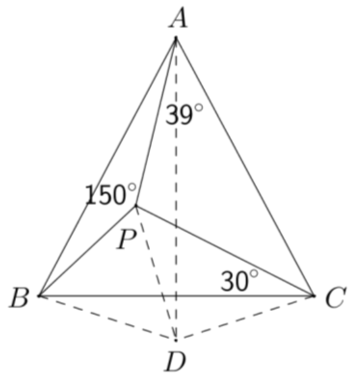
- 3 pure groups — there is only 1 way of grouping
- 2 pure groups — this is not possible (if two groups are pure, so is the third)
- 1 pure group — there are 3 ways of grouping , corresponding to choosing one colour (say, R) out of three to form a pure group (the other two groups must be YYG and YGG)

- no pure group — either every group contains R (2 ways, {RYG, RYG, RYG} or {RYG, RYY, RGG}) , or otherwise the grouping is {RRX, RXX, XXX} where each X is either Y or G; there are two choices for the first X, then two choices for the ‘XX’ in ‘RXX’ (e.g. after RRY we can choose RYG or RGG for the second group, but not RYY), giving a total of $2+2\times 2=6$ ways in this case.

Hence the answer is $1+0+3+6=10$.

9. Let D be the circumcentre of ΔBCP. We have $\angle PDB = 2\angle PCB = 60^\circ$ and $DB = DP$, which means ΔBDP is equilateral and so $\angle APD = 360^\circ - 150^\circ - 60^\circ = 150^\circ = \angle APB$. Together with $PB = PD$, we have $\Delta APB \cong \Delta APD$.

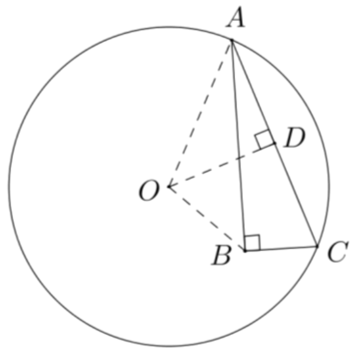
   Next, from $AD = AB = AC$ and $DB = DC$, we get that $\Delta ABD \cong \Delta ACD$. From these, we have $\angle BAP = \angle DAP$ and $\angle BAD = \angle CAD$. It follows that $\angle BAP = \frac{1}{3}\angle CAP = 13^\circ$.



10. Let O be the centre of the circle and let D be the midpoint of AC. Then $OD \perp AC$. By Pythagoras’ Theorem, we have $AC = \sqrt{6^2+2^2} = 2\sqrt{10}$. This gives $AD = \sqrt{10}$ and hence $OD = \sqrt{\sqrt{50}^2 - \sqrt{10}^2} = 2\sqrt{10}$.

    Next, we apply the compound angle formula to obtain

$$\begin{aligned}\cos\angle OAB &= \cos(\angle OAD - \angle BAC)\\ &= \cos\angle OAD\cos\angle BAC + \sin\angle OAD\sin\angle BAC\\ &= \frac{\sqrt{10}}{\sqrt{50}}\times\frac{6}{2\sqrt{10}} + \frac{2\sqrt{10}}{\sqrt{50}}\times\frac{2}{2\sqrt{10}} = \frac{1}{\sqrt{2}}\end{aligned}$$



    It thus follows from the cosine formula that

$$OB = \sqrt{AO^2 + AB^2 - 2AO\times AB\cos\angle OAB} = \sqrt{26}.$$

11. If we take any 46 two-digit numbers, then only 44 two-digit numbers are not chosen. Hence, we either have one of the 36 pairs $\left\{\overline{AB}, \overline{BA}\right\}$ where $1 \le A < B \le 9$, or one of the 9 numbers 11, 22, …, 99. In either case, we get a ‘dragon sequence’ by definition.

    Next, suppose we only pick those two-digit numbers whose unit digit is smaller than the tens digit. There are altogether 45 such numbers. If we get a ‘dragon sequence’ among these numbers, say $\left\{\overline{A_1A_2}, \overline{A_2A_3}, \cdots, \overline{A_nA_1}\right\}$, then $A_1 > A_2 > \cdots > A_n > A_1$, which is a contradiction. So these 45 numbers contain no ‘dragon sequence’. It follows that the answer is 46.

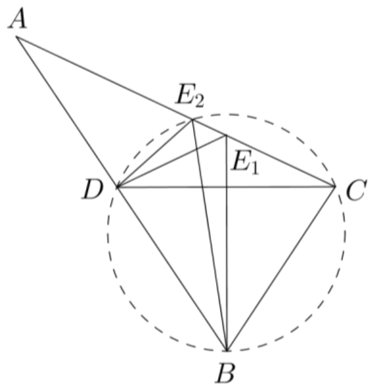
12. Note that the remainder when 2^n is divided by 3 is 2 when n is odd, and is 1 when n is even. Hence $\left[\frac{2^n}{3}\right]=\frac{2^n-2}{3}$ when n is odd, and $\left[\frac{2^n}{3}\right]=\frac{2^n-1}{3}$ if n is even. It follows that

$$\begin{aligned}
S &= \left[\frac{1}{3}\right]+\left[\frac{2}{3}\right]+\left[\frac{2^2}{3}\right]+\cdots+\left[\frac{2^{2014}}{3}\right] \\
&= 0+\left(\frac{2-2}{3}+\frac{2^2-1}{3}\right)+\left(\frac{2^3-2}{3}+\frac{2^4-1}{3}\right)\cdots+\left(\frac{2^{2013}-2}{3}+\frac{2^{2014}-1}{3}\right) \\
&= \left(\frac{2}{3}+\frac{2^2}{3}-1\right)+\left(\frac{2^3}{3}+\frac{2^4}{3}-1\right)+\cdots+\left(\frac{2^{2013}}{3}+\frac{2^{2014}}{3}-1\right) \\
&= \left(\frac{2}{3}+\frac{2^2}{3}+\frac{2^3}{3}\cdots+\frac{2^{2014}}{3}\right)-1007 \\
&= \frac{2^{2015}-2}{3}-1007
\end{aligned}$$

The last two digits of powers of 2 are listed as follows: 02, 04, 08, 16, 32, 64, 28, 56, 12, 24, 48, 96, 92, 84, 68, 36, 72, 44, 88, 76, 52, 04, 08, … The pattern repeats when the exponent is increased by 20. So the last two digits of 2^{2015} are the same as those of 2^{15}, i.e. 68.

Now write $2^{2015}-2=100k+66$. Since $\frac{2^{2015}-2}{3}$ is an integer, k is a multiple of 3, say $k=3m$. Thus the last two digits of S are the same as those of $100m+22-7$, i.e. 15.

13. Obviously one possible position of E arises from the case when $BE\perp CD$ (or equivalently, $ED=EC$). It is denoted by E_1 in the figure, in which case E_1DBC is a kite. In particular, BE_1 bisects $\angle CBA$ and hence $\frac{AE_1}{E_1C}=\frac{AB}{BC}=\frac{13}{7}$.



With b denoting the length of AC, we get $AE_1=\frac{13b}{20}$.

If $ED\neq EC$, then the internal bisector of $\angle DEC$ intersects the perpendicular bisector of CD on the circumcircle of ΔCDE. But we know that this intersection is B and so E must be the second intersection of the circumcircle of ΔBCD with side AC. This is the point E_2 in the figure. The power chord theorem thus gives $AE_2\times AC=AD\times AB$ and so $AE_2=\frac{AD\times AB}{AC}=\frac{(13-7)13}{b}=\frac{78}{b}$.

It follows that the answer is $\frac{13b}{20}\times\frac{78}{b}=\frac{507}{10}$.

Remark. The two cases are the same, i.e. $E_1=E_2$, only if $\angle ACB=90^\circ$, which is excluded since the question assumed that ΔABC is acute. On the other hand, the question intended to mean that ΔABC is *fixed*; indeed the length of AE itself has infinitely many possible values, but with each fixed ΔABC there are only two and the product of them is the same regardless of the choice of ΔABC.

14. The prime factorisation of 2013×2014 is $2\times3\times11\times19\times53\times61$. The problem is the same as counting the number of ways of distributing the 6 primes into 3 groups such that each group contains at least one prime. For example, if the groups are $\{2,3,11\}$, $\{19,53\}$ and $\{61\}$, then the corresponding factors are 66, 1007 and 61. By ordering the factors, we get $a=61$, $b=66$ and $c=1007$. We have a few cases to consider:

- The number of primes in the 3 groups are 4, 1, 1 respectively (we shall abbreviated this as case (4,1,1)) — there are $C_2^6=15$ choices for the two singletons.
- Case (3,2,1) — there are 6 choices for the singleton and then $C_2^5=10$ choices for the doubleton, leading to $6\times10=60$ ways in this case.
- Case (2,2,2) — there are 5 ways to choose a partner for the prime 2, and 3 ways to choose a partner for one of the remaining primes. Afterwards, the two primes left form the last group automatically. Hence there are $3\times5=15$ ways to do so.

It follows that the answer is $15+60+15=90$.

15. Since $x^3-1=(x-1)(x^2+x+1)$, we have $x^3\equiv1$ (mod x^2+x+1). We consider the following three cases (in what follows k is a non-negative integer):

- If $n=3k$, we have $x^{2n}+x^n+1=(x^3)^{2k}+(x^3)^k+1\equiv(1)^{2k}+(1)^k+1=3$ (mod x^2+x+1). Hence x^2+x+1 is a not factor of $x^{2n}+x^n+1$ in this case.
- If $n=3k+1$, we have $x^{2n}+x^n+1=(x^3)^{2k}\cdot x^2+(x^3)^k\cdot x+1\equiv(1)^{2k}\cdot x^2+(1)^k\cdot x+1\equiv0$ (mod x^2+x+1). Hence x^2+x+1 is always a factor of $x^{2n}+x^n+1$ in this case.
- Finally, if $n=3k+2$, we have

  $x^{2n}+x^n+1=(x^3)^{2k}\cdot x^4+(x^3)^k\cdot x^2+1\equiv(1)^{2k}\cdot x^4+(1)^k\cdot x^2+1\equiv x^4+x^2+1$ (mod x^2+x+1).

  According to the previous case, x^4+x^2+1 is divisible by x^2+x+1. That is, x^2+x+1 is always a factor of $x^{2n}+x^n+1$ in this case.

It follows that the answer is

$$(1+2+\cdots+2014)-(3+6+\cdots+2013)=\frac{(1+2014)\times2014}{2}-\frac{(3+2013)\times671}{2}=1352737.$$

Remark. We may begin by trials on small cases which show that 1, 2, 4, 5 are possible values of n while 3 and 6 are not. This would easily have led us to guess the answer correctly, and suggest that in working out a complete solution we should consider the values of n modulo 3.

16. Clearly the maximum value exists as $a_1+2a_2+3a_3+\cdots+24a_{24}\le 1^2+2^2+\cdots+24^2$. Let's see what happens when maximality is reached. We claim that at maximality, we must have $a_m=-m$ or $a_n=n$ for any pair (m, n) satisfying $1\le m<n\le 24$. Indeed, if this is not true for some such (m, n), then we can replace a_m by a_m-1 and a_n by a_n+1, so that the conditions are still satisfied, but the value of $a_1+2a_2+3a_3+\cdots+24a_{24}$ is increased by $(n-m)$. This contradicts the maximality and hence proves the claim.

   In view of the claim and the fact that the sum of a_1 to a_{24} is zero, when maximality is reached there must exist integers s and t such that $a_m=-m$ for $1\le m\le s$ and $a_n=n$ for $t\le n\le 24$, where t can only be equal to $s+1$ or $s+2$, for otherwise we may take $m=s+1$ and $n=t-1$ and the claim is violated.

   - The case $t=s+1$ cannot occur, for otherwise we will get $-(1+2+\cdots s)+[(s+1)+(s+2)+\cdots+24]=0$, or $1+2+\cdots+s=\dfrac{1+2+\cdots+24}{2}=150$, but 150 is not a triangular number.
   - Hence we must have $t=s+2$. We check that the last triangular number below 150 is $1+2+\cdots+16=136$. On the other hand, $18+19+\cdots+24=147$. Hence $a_{17}=-11$. The answer is thus $-(1^2+2^2+\cdots+16^2)+17(-11)+(18^2+19^2+\cdots+24^2)=1432$.

17. Let $x=\sqrt[3]{\sqrt{5}+2}+\sqrt[3]{\sqrt{5}-2}$. Using the formula $(a+b)^3=a^3+b^3+3ab(a+b)$, we have

$$\begin{aligned} x^3 &= (\sqrt{5}+2)+(\sqrt{5}-2)+3\sqrt[3]{(\sqrt{5}+2)(\sqrt{5}-2)}x \\ &= 2\sqrt{5}+3x \end{aligned}$$

   Rewrite this as $(x-\sqrt{5})(x^2+\sqrt{5}x+2)=0$. Since the quadratic has no real roots, we must have $x=\sqrt{5}$. So $\left[\left(\sqrt[3]{\sqrt{5}+2}+\sqrt[3]{\sqrt{5}-2}\right)^{2014}\right]=\left[\sqrt{5}^{2014}\right]=5^{1007}$. The last 3 digits of the powers of 5 follow the pattern 005, 025, 125, 625, 125, 625, … It follows that the answer is 125.

18. For $0\le i\le 35$, let a_i be the number of participants who made exactly i handshakes. For a fixed i, it is given that the corresponding a_i participants did not shake hands with each other. So each of them made at most $36-a_i$ handshakes. In other words, we have $a_i\le 36-i$. Thus the total number of handshakes made is

$$\frac{0\times a_0+1\times a_1+\cdots+35\times a_{35}}{2}=\frac{(a_{35})+(a_{35}+a_{34})+(a_{35}+a_{34}+a_{33})+\cdots+(a_{35}+a_{34}+\cdots+a_1)}{2}.$$

   There are 35 pairs of parentheses on the right hand side. Since $a_i\le 36-i$ we have $a_{35}\le 1$,

$a_{34} \le 2$, etc. On the other hand the sum inside each pair of parentheses is bounded above by the total number of participants, i.e. 36. We thus get the following upper bound on the number of handshakes:

$$\frac{(1)+(1+2)+(1+2+3)+\cdots+(1+2+\cdots+8)+36+36+\cdots+36}{2}$$

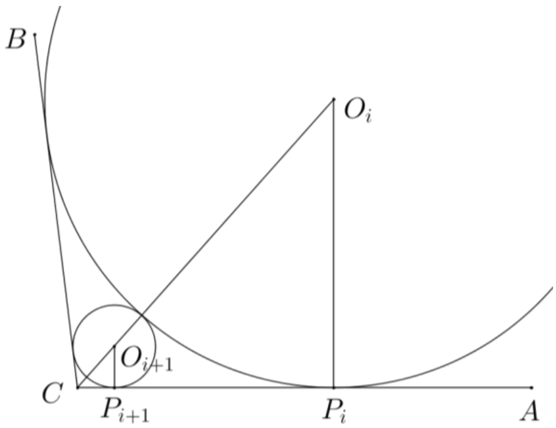
The above expression is equal to $\frac{1+3+6+10+15+21+28+36\times 28}{2}=546$. This upper bound is indeed attainable, as follows: Partition the 36 participants into 8 groups with 1, 2, …, 8 people respectively. Two participants shook hand with each other if and only if they belong to different groups. Clearly the condition in the question is satisfied and the total number of handshakes in this case is

$$C_2^{36}-C_2^2-C_2^3-C_2^4-\cdots-C_2^8=630-1-3-6-10-15-21-28=546,$$

so the answer is 546.

19. Consider two consecutive circles ω_i and ω_{i+1}. Let their centres be O_i and O_{i+1} respectively. Since both circles are tangent to CA and CB, both centres lie on the internal bisector of $\angle ACB$. In other words, C, O_i, O_{i+1} are collinear and $\angle O_iCA = 60°$. Denote by P_i and P_{i+1} the points of tangency of ω_i and ω_{i+1} to CA respectively. Then $O_iP_i \perp CA$ and $O_{i+1}P_{i+1} \perp CA$.

Consider the trapezium $O_iP_iP_{i+1}O_{i+1}$. We have $O_iP_i = r_i$, $O_{i+1}P_{i+1} = r_{i+1}$, $O_iO_{i+1} = r_i + r_{i+1}$ and $P_iP_{i+1} = P_iC - P_{i+1}C = \frac{1}{\sqrt{3}}(r_i - r_{i+1})$. Note that the last equality follows from $\angle O_iCP_i = 60°$.



Using Pythagoras' Theorem, we have

$$\left(\frac{1}{\sqrt{3}}(r_i - r_{i+1})\right)^2 + (r_i - r_{i+1})^2 = (r_i + r_{i+1})^2$$

$$\frac{2}{\sqrt{3}}(r_i - r_{i+1}) = r_i + r_{i+1}$$

This can be simplified to $r_{i+1} = (7-4\sqrt{3})r_i$. Therefore the radii of the circles form a geometric sequence with initial term 3 and common ratio is $7-4\sqrt{3}$. It follows that the answer is

$$\frac{3}{1-(7-4\sqrt{3})} = \frac{3}{2}+\sqrt{3}.$$

20. Firstly, we show that n can be 3721. Since $60^{60} = 2^{120}\times 3^{60}\times 5^{60}$, the factors of 60^{60} are in the form $2^a \times 3^b \times 5^c$ where $0 \le a \le 120$ and $0 \le b, c \le 60$. Suppose all the student numbers take this form and satisfy the condition $a+b+c=120$. Then each student number is uniquely

determined by b and c, so that there are altogether $61^2 = 3721$ students. To show that the condition of the question is satisfied in this case, it suffices to show that no number of this form is a proper factor of the other (since the H.C.F. of two distinct numbers must be a proper factor of at least one of these numbers). Indeed, if $2^a \times 3^b \times 5^c$ is a proper factor of $2^x \times 3^y \times 5^z$, then $a \le x$, $b \le y$ and $c \le z$, with at least one strict inequality. But this is a contradiction, since $120 = a + b + c < x + y + z = 120$. Hence 3721 is a possible value of n.

Now suppose there are at least 3722 student numbers. Since there are only $61^2 = 3721$ choices for the exponents b and c, there are two student numbers having the same pairs of exponents (b, c) by the pigeonhole principle. In that case, the smaller one is a proper factor of the other and hence is the H.C.F. of these two numbers. It follows that the answer is 3721.