香港資優教育學苑
The Hong Kong Academy for Gifted Education

International Mathematical Olympiad Preliminary Selection Contest – Hong Kong 2020

國際數學奧林匹克 — 香港選拔賽初賽 2020

13 Jun 2020 (Saturday)
2020 年 6 月 13 日（星期六）

Question Book
問題簿

Instructions to Contestants:
考生須知：

1. The contest comprises a 3 hours written test.
   比賽以筆試形式進行，限時三小時。

2. Questions are in bilingual versions. Answer all questions.
   題目中英對照。全卷題目均須作答。

3. Put your answers on the answer sheet.
   請將答案寫在答題紙上。

4. The use of calculators is NOT allowed.
   不可使用計算機。

5. Measuring instruments like rulers, compasses, etc. can be used.
   直尺、圓規及其它量度工具可作輔助之用。

Co-organised by The Hong Kong Academy for Gifted Education,
the Gifted Education Section of the Education Bureau and
International Mathematical Olympiad Hong Kong Committee
香港資優教育學苑、教育局資優教育組及國際數學奧林匹克香港委員會合辦

1. Let $n=(10^{2020}+2020)^2$. Find the sum of all the digits of *n*. (1 mark)

設 $n=(10^{2020}+2020)^2$ 。求 *n* 的所有數字之和。（1 分）

2. Let *x*, *y*, *z* be positive integers satisfying $x<y<z$ and $x+xy+xyz=37$. Find the greatest possible value of $x+y+z$. (1 mark)

設 x、y、z 為滿足 $x<y<z$ 和 $x+xy+xyz=37$ 的正整數。求 $x+y+z$ 的最大可能值。（1 分）

3. A child lines up 2020^2 pieces of bricks in a row, and then remove bricks whose positions are square numbers (i.e. the 1st, 4th, 9th, 16th, … bricks). Then he lines up the remaining bricks again and remove those that are in a 'square position'. This process is repeated until the number of bricks remaining drops below 250. How many bricks remain in the end? (1 mark)

一名孩子把 2020^2 塊積木排成一行，然後把位置為平方數的積木（即第 1、4、9、16、…塊積木）移走。之後他又把餘下的積木排成一行，再把位置是平方數的積木移走，然後不斷重複此步驟，直至餘下的積木數目少於 250 為止。那麼，最後餘下多少塊積木？（1 分）

4. In a game, a participant chooses a nine-digit positive integer $\overline{\mathrm{ABCDEFGHI}}$ with distinct non-zero digits. The score of the participant is $\mathrm{A}^{\mathrm{B}^{\mathrm{C}^{\mathrm{D}^{\mathrm{E}^{\mathrm{F}^{\mathrm{G}^{\mathrm{H}^{\mathrm{I}}}}}}}}}$. Which nine-digit number should be chosen in order to maximise the score? (1 mark)

在一個遊戲中，參加者選擇一個由不同的非零數字組成的九位正整數 $\overline{\mathrm{ABCDEFGHI}}$，而其得分為 $\mathrm{A}^{\mathrm{B}^{\mathrm{C}^{\mathrm{D}^{\mathrm{E}^{\mathrm{F}^{\mathrm{G}^{\mathrm{H}^{\mathrm{I}}}}}}}}}$。為使得分達至最大值，參加者應選擇哪一個九位數？（1 分）

5. The 28 students of a class are seated in a circle. They then all claim that 'the two students next to me are of different genders'. It is known that all boys are lying while exactly 3 girls are lying. How many girls are there in the class? (1 mark)

某班 28 名學生圍圈而坐，然後各人均宣稱「我身旁的兩人性別不同」。已知所有男生均在說謊，而說謊的女生則有剛好 3 名。那麼，班中共有多少名女生？（1 分）

6. In ΔABC, $AB=6$, $BC=7$ and $CA=8$. Let D be the mid-point of minor arc AB on the circumcircle of ΔABC. Find AD^2. (1 mark)

在 ΔABC 中，$AB=6$、$BC=7$ 及 $CA=8$。設 D 為 ΔABC 的外接圓上劣弧 AB 的中點。求 AD^2 。（1 分）

7. Solve the equation $\sqrt{7-x}=7-x^2$, where $x>0$. (1 mark)

解方程 $\sqrt{7-x}=7-x^2$ ，其中 $x>0$ 。（1 分）

8. Find the smallest positive multiple of 77 whose last four digits (from left to right) are 2020. (1 mark)

求 77 的最小正倍數，使其最後四位數字（從左至右）為 2020。（1 分）

9. In ΔABC, $\angle B = 46.6°$. D is a point on BC such that $\angle BAD = 20.1°$. If $AB = CD$ and $\angle CAD = x°$, find x. (1 mark)

在 ΔABC 中，$\angle B = 46.6°$。D 是 BC 上的一點，使得 $\angle BAD = 20.1°$。若 $AB = CD$，且 $\angle CAD = x°$，求 x。（1 分）

10. Let k be an integer. If the equation $(x-1)|x+1| = x + \frac{k}{2020}$ has three distinct real roots, how many different possible values of k are there? (1 mark)

設 k 為整數。若方程 $(x-1)|x+1| = x + \frac{k}{2020}$ 有三個不同的實根，則 k 有多少個不同的可能值？（1 分）

11. Let a, b, c be the three roots of the equation $x^3-(k+1)x^2+kx+12=0$, where k is a real number. If $(a-2)^3+(b-2)^3+(c-2)^3=-18$, find the value of k. (2 marks)

設 a、b、c 為方程 $x^3-(k+1)x^2+kx+12=0$（其中 k 為實數）的三個根。若 $(a-2)^3+(b-2)^3+(c-2)^3=-18$，求 k 的值。（2 分）

12. There are some balls, on each of which a positive integer not exceeding 14 (and not necessarily distinct) is written, and the sum of the numbers on all balls is S. Find the greatest possible value of S such that, regardless of what the integers are, one can ensure that the balls can be divided into two piles so that the sum of the numbers on the balls in each pile does not exceed 129. (2 marks)

現有一些球，每個球上均寫有一個不超過 14 的正整數（且可以重複），而所有球上的整數之和為 S。求 S 的最大值，使得不論球上所寫的整數是甚麼，這些球都總可以分成兩堆，當中每堆球上的整數之和均不超過 129。（2 分）

13. There are n different integers on the blackboard. Whenever two of these integers are chosen, either their sum or their difference (possibly both) will be a positive integral power of 2. Find the greatest possible value of n. (2 marks)

黑板上有 n 個互不相同的整數。在當中任意選出兩數，則兩數之和或是兩數之差（也可能和差亦然）必為 2 的正整數次方。求 n 的最大可能值。（2 分）

14. In ΔABC, $\angle ABC = 120°$. The internal bisector of $\angle B$ meets AC at D. If $BD = 1$, find the smallest possible value of $4BC + AB$. (2 marks)

在 ΔABC 中，$\angle ABC = 120°$。$\angle B$ 的內角平分線交 AC 於 D。若 $BD = 1$，求 $4BC + AB$ 的最小可能值。（2 分）

15. How many ten-digit positive integers consist of ten different digits and are divisible by 99? (2 marks)

有多少個十位正整數的十個數字互不相同，且可被 99 整除？（2 分）

16. ΔABC is right-angled at B, with $AB = 1$ and $BC = 3$. E is the foot of perpendicular from B to AC. BA and BE are produced to D and F respectively such that D, F, C are collinear and $\angle DAF = \angle BAC$. Find the length of AD. (2 marks)

ABC 是直角三角形，其中 B 是直角，且 $AB = 1$、$BC = 3$。E 是 B 到 AC 的垂足。BA 和 BE 分別延長至 D 和 F，使得 D、F、C 成一直線，且 $\angle DAF = \angle BAC$。求 AD 的長度。（2 分）

17. How many positive integer solutions does the following system of equations have? (2 marks)

以下方程組有多少個正整數解？（2 分）

$$\begin{cases} \sqrt{2020}(\sqrt{a}+\sqrt{b}) = \sqrt{(c+2020)(d+2020)} \\ \sqrt{2020}(\sqrt{b}+\sqrt{c}) = \sqrt{(d+2020)(a+2020)} \\ \sqrt{2020}(\sqrt{c}+\sqrt{d}) = \sqrt{(a+2020)(b+2020)} \\ \sqrt{2020}(\sqrt{d}+\sqrt{a}) = \sqrt{(b+2020)(c+2020)} \end{cases}$$

18. Two n-sided polygons are said to be of the same type if we can label their vertices in clockwise order as $A_1, A_2, \ldots, A_n$ and $B_1, B_2, \ldots, B_n$ respectively such that each pair of interior angles A_i and B_i are either both reflex angles or both non-reflex angles. How many different types of 11-sided polygons are there? (2 marks)

如果兩個 n 邊形的頂點可分別依順時針方向記為 A_1、A_2、⋯、A_n 及 B_1、B_2、⋯、B_n，使得每對內角 A_i 和 B_i 要麼同時是反角，要麼同時不是反角，則我們稱這兩個 n 邊形為同類。那麼，共有多少種不同類的 11 邊形？（2 分）

19. Four couples are to be seated in a row. If it is required that each woman may only sit next to her husband or another woman, how many different possible seating arrangements are there? (2 marks)

現要把四對夫婦排成一行而坐，並要求每位女士的隔壁只能是她的丈夫或另一位女士。那麼，安排座位的方法共有多少種？（2 分）

20. Consider the Fibonacci sequence 1, 1, 2, 3, 5, 8, 13, … What are the last three digits (from left to right) of the 2020th term? (2 marks)

考慮斐波那契數列 1, 1, 2, 3, 5, 8, 13, …。第 2020 項的最後三位數字（從左至右）是甚麼？（2 分）

– End of Paper – 全卷完 –