International Mathematical Olympiad
Hong Kong Preliminary Selection Contest 2021

Outline of solutions

Answers:

1.	18	**2.**	816	**3.**	$\frac{7}{3}$	**4.**	$\frac{144}{15625}$
5.	674	**6.**	$10\sqrt{13}$	**7.**	$\frac{45}{8}$	**8.**	2049
9.	$\frac{2013}{2041210}$	**10.**	1980	**11.**	$-\frac{10}{11}$	**12.**	107.25
13.	1741	**14.**	14630	**15.**	$\frac{17}{20}$	**16.**	1572
17.*	$\frac{24\sqrt{7}}{7}$	**18.**	66°	**19.**	99	**20.**	$\frac{45}{2}$

*This question was cancelled in the contest due to a misprint in the original question paper.

Solutions:

1. As there are 11 students whose student number is greater than 89 but only one such number is prime (97), at least two group leaders have a student number that is not prime. On the other hand, it is easy to construct a grouping in which all but two group leaders have prime student numbers, essentially via a 'greedy algorithm' (those underlined are group leaders whose student number is prime):

100, 99, 98, 96, 95	94, 93, 92, 91, 90	<u>**97**</u>, 88, 87, 86, 85	<u>**89**</u>, 84, 82, 81, 80
<u>**83**</u>, 78, 77, 76, 75	<u>**79**</u>, 74, 72, 70, 69	<u>**73**</u>, 68, 66, 65, 64	<u>**71**</u>, 63, 62, 60, 58
<u>**67**</u>, 57, 56, 55, 54	<u>**61**</u>, 52, 51, 50, 49	<u>**59**</u>, 48, 46, 45, 44	<u>**53**</u>, 42, 40, 39, 38
<u>**47**</u>, 36, 35, 34, 33	<u>**43**</u>, 32, 30, 28, 27	<u>**41**</u>, 26, 25, 24, 22	<u>**37**</u>, 21, 20, 18, 16
<u>**31**</u>, 15, 14, 12, 10	<u>**29**</u>, 9, 8, 6, 4	<u>**23**</u>, 19, 17, 13, 11	<u>**7**</u>, 5, 3, 2, 1

It follows that the answer is 18.

2. Clearly, no digit of n can be 0. If the hundreds digit of n is 9, then n is divisible by 9. As 999 is not divisible by $9 \times 9 \times 9$, the sum of digits of n must be 18, meaning that the unit digit of n is even (if the unit digit is odd, then the tens digit would be even, in which case n would be odd and cannot be divisible by the product of its digits). Hence we check 972, 954, 936 and 918, and none of these satisfies the required property.

We then consider the case where the hundreds digit of n is 8. Then n is divisible by 8, so the unit digit of n is even. Thus the product of the digits of n, and hence n, must be divisible by 16. We check such numbers in descending order: 896, 864, 848, 832, 816, and only 816 satisfies the requirement. It follows that the greatest possible value of n is 816.

3. Multiplying both equations by ab, we get

$$a^2 + b^2 = 5ab \quad \text{and} \quad a^3 + b^3 = 12ab$$

respectively. Let $a + b = S$ and $ab = P$. Since $a^2 + b^2 = (a + b)^2 - 2ab$ and $a^3 + b^3 = (a + b)^3 - 3ab(a + b)$, the above equations can be rewritten as

$$S^2 - 2P = 5P \quad \text{and} \quad S^3 - 3PS = 12P$$

respectively. The former gives $S^2 = 7P$, which, when plugged in to the latter, gives $7PS - 3PS = 12P$. As $P \neq 0$, we get $S = 3$ and hence $P = \frac{9}{7}$. It follows that

$$\frac{1}{a} + \frac{1}{b} = \frac{S}{P} = \frac{7}{3}.$$

4. As there are 25 primes less than 100, there are altogether $25^3 = 15625$ possible outcomes. Since $2021 = 43 \times 47$, the teacher's product is divisible by 2021 if and only if

- one student chooses 43, one chooses 47 and the other chooses one of the other 23 primes ($23 \times 3! = 138$ possibilities);
- two students choose 43 and one student chooses 47 (3 possibilities for permutation); or
- two students choose 47 and one student chooses 43 (3 possibilities for permutation).

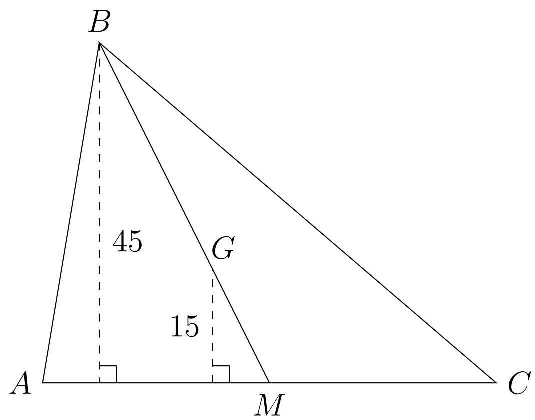
Hence the answer is $\dfrac{138 + 3 + 3}{15625} = \dfrac{144}{15625}$.

5. Let S be the sum of the three numbers recorded. For each question, either all three participants choose the same answer (in which case the question will contribute 3 to S), or two participants choose the same answer and the third chooses a different answer (in which case the question will contribute 1 to S). Hence S is at least 2021, and so at least one the numbers recorded exceeds 673 (for $673 \times 3 < 2021$), i.e. the score of the team is at least 674. Now consider the following scenario.

	Ann's answer	Ben's answer	Cat's answer
First 674 questions	Yes	Yes	No
Next 674 questions	Yes	No	Yes
Last 673 questions	No	Yes	Yes

In the above example Ann and Ben chose the same answer in 674 questions, Ann and Cat chose the same answer in 674 questions while Ben and Cat chose the same answer in 673 questions. In this case the score of the team is exactly 674. It follows that the minimum possible score is 674.

6. Let M be the mid-point of AC. Since $BM = 3GM$, the distance from B to AC is three times the distance from G to AC, i.e. $15 \times 3 = 45$.



Similarly, the altitudes from A and C have lengths $12 \times 3 = 36$ and $20 \times 3 = 60$ respectively. As the length of an altitude is inversely proportional to the corresponding base, we have

$$AB : AC : BC = \frac{1}{60} : \frac{1}{45} : \frac{1}{36} = 3 : 4 : 5.$$

It follows that $\triangle ABC$ is right-angled at A with $AB = 45$ (same as the distance from B to AC). Similarly $AC = 60$ and hence $AM = 30$, and so we have $BM = \sqrt{45^2 + 30^2} = 15\sqrt{13}$ and thus $BG = \frac{2}{3}BM = 10\sqrt{13}$.

7. Let $y = 2\sqrt[5]{x+1} - 2$. The equation thus becomes $(y+1)^4 + (y-1)^4 = 16$, which simplifies to $2(y^2-1)(y^2+7) = 0$. For x to be real, we must have $y = \pm 1$.

- If $y = 1$, we have $\sqrt[5]{x+1} = \dfrac{3}{2}$ and hence $x = \dfrac{211}{32}$.
- If $y = -1$, we have $\sqrt[5]{x+1} = \dfrac{1}{2}$ and hence $x = -\dfrac{31}{32}$.

The answer is thus $\dfrac{211}{32} + \left(-\dfrac{31}{32}\right) = \dfrac{45}{8}$.

8. Note that if an odd number is the sum of two non-negative integral powers of 2, then it must be of the form $2^a + 2^0$ for some positive integer a. Hence we have the following two cases.

- If n is odd, then so is $n+2$, and hence $n = 2^a + 2^0$ and $n + 2 = 2^b + 2^0$ for some positive integers a and b. This implies $2^b = 2^a + 2$, so $(a, b) = (1, 2)$ and hence $n = 3$. We check that this works as $n = 2^1 + 2^0$, $n + 1 = 2^1 + 2^1$ and $n + 2 = 2^2 + 2^1$.
- If n is even, then $n + 1$ is odd and so $n + 1 = 2^a + 2^0$ for some positive integer a, i.e. $n = 2^a$. All such n not exceeding 2021 will work as we have $n = 2^{a-1} + 2^{a-1}$, $n + 1 = 2^a + 2^0$ and $n + 2 = 2^a + 2^1$.

It follows that the answer is $3 + 2 + 4 + 8 + \cdots + 1024 = 2049$.

9. For a favourable outcome, the two balls should have numbers k and $2^n k$ for some positive integers k and n. Since $2^n k$ cannot exceed 2021, for each fixed choice of n, the value of k can range from 1 to the greatest integer not exceeding $\frac{2021}{2^n}$. It follows that the number of favourable outcomes is

$$\left\lfloor \frac{2021}{2^1} \right\rfloor + \left\lfloor \frac{2021}{2^2} \right\rfloor + \left\lfloor \frac{2021}{2^3} \right\rfloor + \cdots = 2013$$

(see remark below). Consequently the answer is

$$\frac{2013}{\binom{2021}{2}} = \frac{2013}{2041210}.$$

Remark. In computing the number of favourable outcomes, the apparently infinite sum is finite since each summand becomes zero once the denominator exceeds 2021. While it is not hard to evaluate the sum directly, it can be shown that the value of this sum is actually equal to 2021 minus the sum of digits in the binary representation of 2021 (in this case the sum is 8 since $2021_{10} = 11111100101_2$). One may also note (though unrelated to this question) that the value of the sum being 2013 means that 2^{2013} is the highest power of 2 that divides 2021!.

10. Suppose the question is of the form $\overline{AB} + \overline{CD}$ where $\overline{AB}$ and $\overline{CD}$ are two-digit numbers. Then it amounts to choosing (A, B, C, D) such that $A, C \in \{1, 2, 3, ..., 9\}$, $B, D \in \{0, 1, 2, ..., 9\}$, $A + C \leq 9$ and $B + D \leq 9$.

    - A may range from 1 to 8, and for each value of A, there are $9 - A$ choices of C for which $A + C \leq 9$. Hence there are $8 + 7 + 6 + \cdots + 1 = 36$ possible sets of values for (A, C).
    - B may range from 0 to 9, and for each value of B, there are $10 - B$ choices of D for which $B + D \leq 9$. Hence there are $10 + 9 + 8 + \cdots + 1 = 55$ possible sets of values for (B, D).

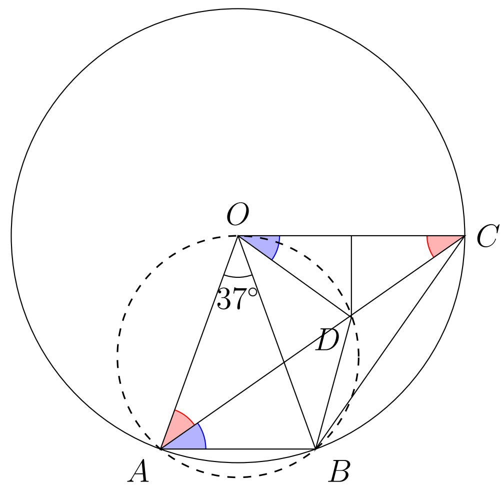
    It follows that the answer is $36 \times 55 = 1980$.

11. The equation can be rewritten as

$$(21x + 20y)^{2021} + 22(21x + 20y) = -x^{2021} - 22x,$$

or $f(21x+20y) = -f(x)$ if we let $f(t) = t^{2021}+22t$. Note that f is odd (i.e. $f(-t) = -f(t)$) and strictly increasing (so that $f(s) = f(t)$ implies $s = t$). We can thus deduce from $f(21x + 20y) = -f(x)$ that $21x + 20y = -x$, and so $\dfrac{x}{y} = -\dfrac{10}{11}$.

12. Let $\angle OAC = \angle OCA = \theta$. Then $\angle DOC$ and $\angle CAB$ are also equal to θ. Now we have $\angle ODA = \angle DOC + \angle DCO = 2\theta$ and $\angle OBA = \angle OAB = 2\theta$, which shows that the points O, A, B, D are concyclic.



It follows that $\angle OBD = \angle OAD = \theta$, so $\angle ABD = 3\theta$. To find the value of θ, consider the interior angles of $\triangle OAB$, from which we get $37^\circ + 4\theta = 180^\circ$, so $\theta = 35.75^\circ$ and so $x = 3(35.75) = 107.25$.

13. We need $n^3 \equiv 2021 \pmod{10000}$, which is equivalent to $n^3 \equiv 5 \pmod{16}$ and $n^3 \equiv 146 \pmod{625}$. We work out the remainders when n is divided by 16 and 625 as follows.

- From $n^3 \equiv 5 \pmod{16}$ we have $n^3 \equiv 1 \pmod 4$ and hence $n \equiv 1 \pmod 4$. Setting $n = 4a + 1$, we have

$$5 \equiv n^3 = (4a+1)^3 = 16(4a^3 + 3a^2) + 12a + 1 \pmod{16},$$

which gives $12a \equiv 4 \pmod{16}$ and hence $3a \equiv 1 \pmod 4$. This gives $a \equiv 3 \pmod 4$ and hence $n \equiv 13 \pmod{16}$.

- Clearly we have $n \equiv 1 \pmod 5$. We also have $n^3 \equiv 146 \equiv -4 \pmod{25}$. Checking 1^3, 6^3, 11^3, 16^2 and 21^3, only $16^3 = 4096$ is also congruent to -4 modulo 25. This shows that $n \equiv 16 \pmod{25}$. Setting $n = 25b + 16$, we have

$$146 \equiv n^3 = (25b+16)^3 = 625(25b^3 + 48b^2) + 19200b + 4096 \pmod{625},$$

which simplifies to $450b \equiv 425 \pmod{625}$, or $18b \equiv 17 \pmod{25}$, or $-7b \equiv 17 \pmod{25}$. Noting that $17 + 25 = (-7)(-6)$, we have $b \equiv -6 \equiv 19 \pmod{25}$, and hence $n \equiv 25(19) + 16 \equiv 491 \pmod{625}$.

Now we need to find the smallest positive integer n for which $n \equiv 13 \pmod{16}$ and $n \equiv 491 \pmod{625}$. The latter means n is one of 491, 1116, 1741, 2366, ..., and by taking modulo 16 on this sequence we find that 1741 is the first term which is congruent to 13 modulo 16. It follows that the smallest such n is 1741.

Remark. It may be more intuitive (though also a bit more tedious) to work out the digits of n one by one. Clearly the unit digit of n is 1. Setting $n = 10a + 1$, we have

$$n^3 = (10a + 1)^3 = 10(100a^3 + 30a^2 + 3a) + 1,$$

so that $3a$ must have unit digit 2 in order for n^3 to have tens digit 2. It follows that a has unit digit 4, so we may write $n = 100b + 41$ and continue in a similar manner to find the hundreds and thousands digits of n.

14. Clearly, student 1 and student 21 never give presentations. For each student numbered n, where $2 \leq n \leq 20$, we count the number of times the student presents. The student will give presentation in a lesson if and only one of the other two students responsible for preparing has number less than n and the other has number larger than n. There are thus $(n - 1)(21 - n)$ such combinations, and this is also the number of times student number n gives presentations as any three students have been preparing notes together exactly once. Hence n will occur in the sum $(n - 1)(21 - n)$ times for each such n. The required sum is thus equal to

$$2 \cdot 1 \cdot 19 + 3 \cdot 2 \cdot 18 + 4 \cdot 3 \cdot 17 + \cdots + 19 \cdot 18 \cdot 2 + 20 \cdot 19 \cdot 1.$$

We can pair up the first and last terms, the second and second last terms, and so on, to get the answer

$$22 \times (1 \cdot 19 + 2 \cdot 18 + 3 \cdot 17 + \cdots + 9 \cdot 11) + 11 \cdot 10 \cdot 10 = 22 \times 615 + 1100 = 14630.$$

Remark. While it is not too hard to evaluate the sum $1 \cdot 19 + 2 \cdot 18 + 3 \cdot 17 + \cdots + 9 \cdot 11$ directly, there is a more systematic way to do so:

$$\begin{aligned} 1 \cdot 19 + 2 \cdot 18 + \cdots + 9 \cdot 11 &= (10^2 - 9^2) + (10^2 - 8^2) + \cdots + (10^2 - 1^2) \\ &= 10^2 \cdot 9 - (1^2 + 2^2 + \cdots + 9^2) \\ &= 900 - \frac{9(9 + 1)(2 \cdot 9 + 1)}{6} \end{aligned}$$

and this alternative method will work better if the parameter 21 in the question is significantly enlarged.

15. Let n be the common digit, a be the other digit in the numerator and b be the other digit in the denominator, where $a < b$ and a, b are relatively prime. Since

$$\frac{10a + n}{10b + n} > \frac{a}{b} \quad \text{and} \quad \frac{10n + a}{10n + b} > \frac{a}{b}$$

whenever $a < b$, the common digit n cannot occur as the unit digit in both the numerator and denominator, nor as the tens digit in both. We consider the following two cases.

- If n occurs as the tens digit in the numerator and unit digit in the denominator, then we have
$$\frac{10n+a}{10b+n}=\frac{a}{b},$$
which simplifies to $10bn=9ab+an$. As $a<b$ we get $10bn<9ab+bn$, which implies $n<a$, but this leads to the contradiction
$$an=10bn-9ab\leq 10b(a-1)-9ab=(a-10)b<0.$$
Hence there is no solution in this case.

- If n occurs as the unit digit in the numerator and tens digit in the denominator, then we have
$$\frac{10a+n}{10n+b}=\frac{a}{b},$$
which simplifies to $10an=9ab+bn$. As $a<b$ we get $10an>9ab+an$, which implies $n>b$. Also, taking modulo 9 on the equality gives $an\equiv bn \pmod 9$. Knowing that $a<b<n$, either $n=9$ or we must have $(a,b,n)=(1,4,6)$ or $(a,b,n)=(2,5,6)$. The last two cases both work as
$$\frac{16}{64}=\frac{1}{4}\quad\text{and}\quad\frac{26}{65}=\frac{2}{5}.$$
It remains to consider $n=9$. The equality becomes $90a=9ab+9b$, or
$$(a+1)(10-b)=10.$$
Given that $1\leq a<b<n=9$, the factor $a+1$ may be 2 or 5, but the latter is rejected since it gives $(a,b)=(4,8)$ which contradicts a and b beng relatively prime. The former gives $(a,b)=(1,5)$ which works as $\dfrac{19}{95}=\dfrac{1}{5}$.

It follows that the answer is $\dfrac{1}{4}+\dfrac{2}{5}+\dfrac{1}{5}=\dfrac{17}{20}$.

16. Let $f(a,b,c)=a^3+b^3+c^3-3abc$. Note that $f(a,b,c)$ can be factorised as
$$(a+b+c)(a^2+b^2+c^2-ab-bc-ca)=\frac{(a+b+c)[(a-b)^2+(b-c)^2+(c-a)^2]}{2}.$$
From this we readily see that
$$f(a,a,a+1)=3a+1\quad\text{and}\quad f(a,a+1,a+1)=3a+2$$
for any a, meaning that all positive integers that are not divisible by 3 can be expressed in the desired form. Now suppose $f(a,b,c)=(a+b+c)(a^2+b^2+c^2-ab-bc-ca)$ is divisible by 3. Note that
$$\begin{aligned}a^2+b^2+c^2-ab-bc-ca&=(a+b+c)^2-3(ab+bc+ca)\\&\equiv(a+b+c)^2\pmod 3.\end{aligned}$$

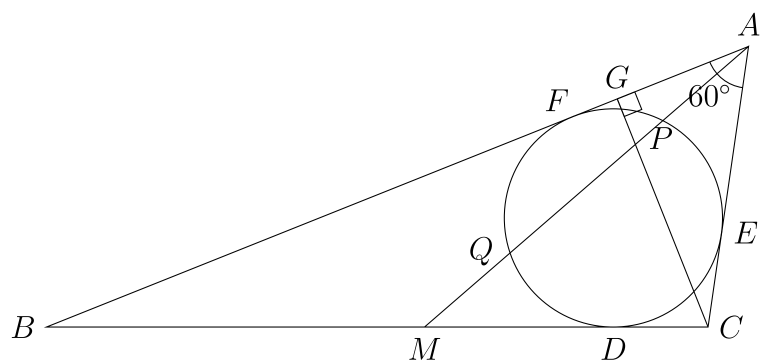
It follows that $a+b+c$, and hence $a^2+b^2+c^2-ab-bc-ca$, must be divisible by 3, in which case $f(a,b,c)$ must be divisible by 9. Finally, we have

$$f(a,a+1,a+2)=9(a+1)$$

and so all positive multiples of 9 can be expressed in the desired form too. Therefore the only positive integers that cannot be expressed in the desired form are those divisible by 3 but not 9, and so the answer is

$$2021-\left(\left\lfloor\frac{2021}{3}\right\rfloor-\left\lfloor\frac{2021}{9}\right\rfloor\right)=1572.$$

17. We have $AF=AE=4$ and $CD=CE=2$. Let $BD=BF=x$, and G be the foot of perpendicular from C to AB. Then $CG=6\sin 60^\circ=3\sqrt{3}$ and $AG=6\cos 60^\circ=3$. It follows that $GF=1$.



Applying Pythagoras' Theorem in $\triangle BCG$, we have $(x+2)^2=(x+1)^2+(3\sqrt{3})^2$, which gives $x=12$. By the power chord theorem (see remark below) and the given fact that $AP=QM$, we have

$$MD^2=MQ\times MP=AP\times AQ=AE^2=16$$

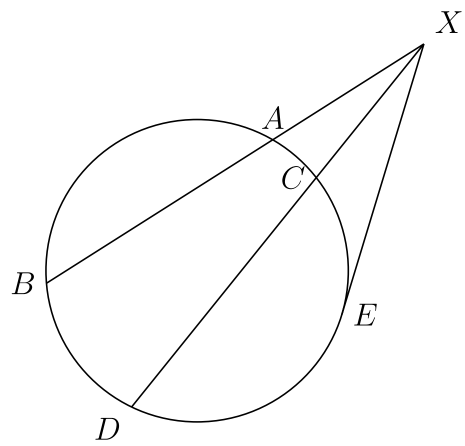
and so $MD=4$. Hence we have $AC=MC=6$. Applying the cosine formula in $\triangle ABC$, we have

$$\cos C=\frac{a^2+b^2-c^2}{2ab}=\frac{14^2+6^2-16^2}{2(14)(6)}=-\frac{1}{7}$$

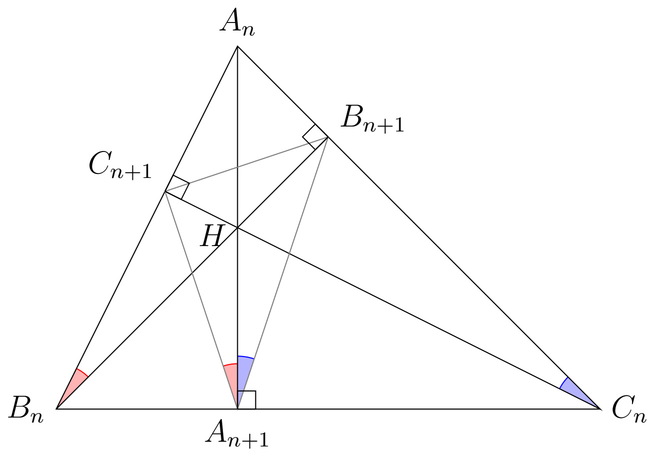
and so applying the formula again in $\triangle AMC$ gives the answer

$$AM=\sqrt{6^2+6^2-2(6)(6)\left(-\frac{1}{7}\right)}=\frac{24\sqrt{7}}{7}.$$

Remark. Let X be a point outside a circle. The power chord theorem asserts that for any straight line passing through X and intersecting the circle at A and B (possibly $A=B$ if the line is tangent to the circle), the value of $XA\cdot XB$ is a constant. Equivalently, we have $XA\cdot XB=XC\cdot XD=XE^2$ in the figure below. The theorem can be easily proved using the fact that $\triangle XAC\sim\triangle XDB$ and $\triangle XEA\sim\triangle XBE$.



18. Suppose $\triangle A_nB_nC_n$ is acute with orthocentre H.



Then $HA_{n+1}B_nC_{n+1}$ and $HA_{n+1}C_nB_{n+1}$ are both cyclic quadrilaterals, so we have

$$\angle HA_{n+1}C_{n+1} = \angle HB_nC_{n+1} = 90^\circ - \angle A_n$$

and similarly $\angle HA_{n+1}B_{n+1} = 90^\circ - \angle A_n$. It follows that $\angle A_{n+1} = 180^\circ - 2\angle A_n$. By writing $\angle A_n = (60 + a_n)^\circ$, this becomes $60 + a_{n+1} = 180 - 2(60 + a_n)$, or $a_{n+1} = -2a_n$. By defining b_n and c_n analogously, we can obtain $b_{n+1} = -2b_n$ and $c_{n+1} = -2c_n$ in the same manner. Note that $a_n + b_n + c_n = 0$ for all n.

Now the game ends when any one of a_n, b_n and c_n becomes 30 or larger. As a_0, b_0 and c_0 are not all zeros and cannot be all positive, at least one of $a_5 = -32a_0$, $b_5 = -32b_0$ and $c_5 = -32c_0$ will be greater than 30 if the game has not yet ended when $n = 4$. As Ben is the one who constructs $\triangle A_5B_5C_5$, Ann must seek to ensure $\triangle A_4B_4C_4$ is not acute for the greatest positive return. Hence one of $a_4 = 16a_0$, $b_4 = 16b_0$ and $c_4 = 16c_0$ must be greater than or equal to 30, whereas all of $a_3 = -8a_0$, $b_3 = -8b_0$ and $c_3 = -8c_0$ must be less than 30.

Subject to the above constraints, we would like to maximise a_0 (hence minimise $b_0 + c_0$). As b_3 and c_3 are to be less than 30, each of b_0 and c_0 can be no less than -3. It follows that a_0 cannot exceed 6. Now we check that $(a_0, b_0, c_0) = (6, -3, -3)$ works as the triangles would evolve as follows:

n	0	1	2	3	4
$\angle A_n$	$66°$	$48°$	$84°$	$12°$	$156°$
$\angle B_n$	$57°$	$66°$	$48°$	$84°$	$12°$
$\angle C_n$	$57°$	$66°$	$48°$	$84°$	$12°$

It follows that the greatest possible value of $\angle A_0$ is $66°$.

19. Since a square can only be congruent to 0 or 1 modulo 3, $a^2 + b^2$ is divisible by 3 if and only if both a^2 and b^2, and hence both a and b, are divisible by 3. It follows that $f(n) = 0$ if n is divisible by 3 but not 9. The desired sum can thus be reduced to

$$f(9) + f(18) + f(27) + \cdots + f(2016).$$

Furthermore, we have $f(9k) = f(k)$, since if $9k = a^2 + b^2$ then both a and b are divisible by 3, so setting $a = 3c$ and $b = 3d$ reduces the equation to $k = c^2 + d^2$, meaning that each way of expressing $9k$ as the sum of two squares corresponds to a way of expressing k as the sum of two squares (and the converse is clearly true). The desired sum is thus further reduced to

$$f(1) + f(2) + f(3) + \cdots + f(224)$$

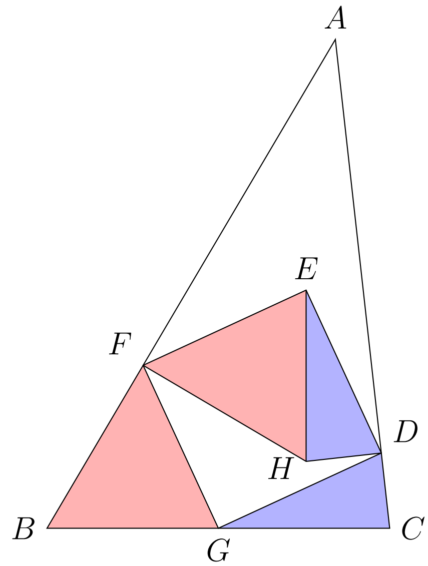
and this is simply equal to the number of pairs of non-negative integers (x, y) for which $x \le y$ and $1 \le x^2 + y^2 \le 224$. We have $0 \le x \le 10$, and for each fixed x the value of y can range from x to $\lfloor \sqrt{224 - x^2} \rfloor$ for a total of $\lfloor \sqrt{224 - x^2} \rfloor - x + 1$ choices. We tabulate this number as follows:

Value of x	0	1	2	3	4	5	6	7	8	9	10
Choices for y	15	14	13	12	11	10	8	7	5	3	2

Not forgetting that the pair $(x, y) = (0, 0)$ has to be rejected, the answer is

$$15 + 14 + 13 + 12 + 11 + 10 + 8 + 7 + 5 + 3 + 2 - 1 = 99.$$

20. With reference to the figure below, we rotate $\triangle GFB$ about F by $90°$ and $\triangle GDC$ about D by $90°$, both towards the interior of $\triangle ABC$. Since $DEFG$ is a square, the image of G under both rotations is E. Furthermore, since $GB = GC$ and $\angle FGB + \angle DGC = 90°$, the images of B and C under the respective rotations are the same, which we label as H.



Note that HF is perpendicular to AB and HD is perpendicular to AC. It follows that

$$AF^2 + FB^2 = AF^2 + FH^2 = AH^2 = AD^2 + DH^2 = 11^2 + 2^2 = 125.$$

Together with $AF + FB = 15$ and $AF > FB$ we can solve to get $AF = 10$ and $FB = 5$. On the other hand, since $\angle FHD = \angle FHE + \angle DHE = \angle B + \angle C = 180° - \angle A$, we see that $AFHD$ is a cyclic quadrilateral. Ptolemy's theorem (see remark below) thus asserts that $AH \cdot DF = AF \cdot HD + AD \cdot FH$, or

$$\sqrt{125} \cdot DF = 10 \cdot 2 + 11 \cdot 5.$$

From this we get $DF = \dfrac{75}{\sqrt{125}} = \sqrt{45}$, and the area of $DEFG$ is thus $\dfrac{1}{2}DF^2 = \dfrac{45}{2}$.

Remark. Ptolemy's theorem asserts that if $ABCD$ is a cyclic quadrilateral, then

$$AC \cdot BD = AB \cdot CD + BC \cdot AD.$$

More generally, we have *Ptolemy's inequality*, which asserts that in any convex quadrilateral $ABCD$,

$$AC \cdot BD \leq AB \cdot CD + BC \cdot AD,$$

and equality holds if and only if $ABCD$ is cyclic. Many different proofs that make use of various tools in algebra and geometry are known.