International Mathematical Olympiad

Preliminary Selection Contest 2019 — Hong Kong

Outline of Solutions

Answers:

1.	11101	**2.**	7560	**3.**	$\frac{10\pi}{3}$	**4.**	12
5.	300	**6.**	$6-\sqrt{15}$	**7.**	160	**8.**	3834
9.	$\sqrt{7}-2$	**10.**	$2+\sqrt{5}$	**11.**	$14:11$	**12.**	$\frac{1}{14}$
13.	$\frac{25}{4}$	**14.**	$\frac{9}{4}$	**15.**	74	**16.**	240
17.	29702	**18.**	$\frac{55}{3}$	**19.**	13	**20.**	$\frac{1}{420}$

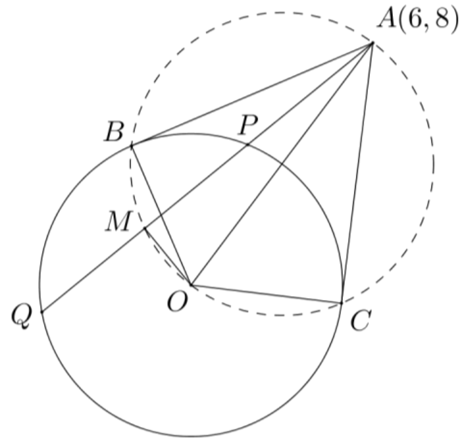
Solutions:

1. We have $1111^{2019} \equiv (-10000)^{2019} = -(10^4)^{2019} = -10^{8076} \pmod{11111}$. As $10^5 = 11111\times 9+1$, we have $1111^{2019} \equiv -10(10^5)^{1615} \equiv -10(1)^{1615} \equiv 11101 \pmod{11111}$. That means the remainder is 11101.

   Remark. Obviously one can also find the answer by observing the pattern of remainders when 1111, 1111^2, 1111^3, … are divided by 11111, although it would be more tedious — the remainders repeat every 10 terms.

2. There are 4 red and 5 green cards. We first choose 5 out of the 9 positions to put the green cards, and there are $C_5^9 = 126$ ways to do so. (The remaining 4 positions must be for the red cards.) Note that the positions of the red cards are then uniquely determined as they must be arranged in ascending order from left to right. On the other hand, we can permute the 5 green cards in $5!\div 2 = 60$ ways (since the two cards with the number 2 are indistinguishable). It follows that the answer is $126\times 60 = 7560$.

3. The circle has centre $O(0,0)$ and radius 5. Let A be the point $(6,8)$ and B, C be the points where the tangents from A to the circle touch the circle.



Since M is the midpoint of the chord PQ, we have $\angle OMA = 90°$. As we also have $\angle OBA = \angle OCA = 90°$, the circle with OA as diameter passes through B, M and C. Hence the locus of M is the minor arc of that circle from B to C. (It is easy to see that every point on the minor arc is a possible position of M.)

Now we have $OB = 5$ and $OA = \sqrt{6^2 + 8^2} = 10$. As $OA = 2OB$ and $\angle OBA = 90°$, we have $\angle BAO = 30°$. It follows that $\angle BAC = 60°$ and so the minor arc BC subtends an angle of 120° at the centre. The length of the locus of M is thus $10\pi \times \frac{120}{360} = \frac{10\pi}{3}$.

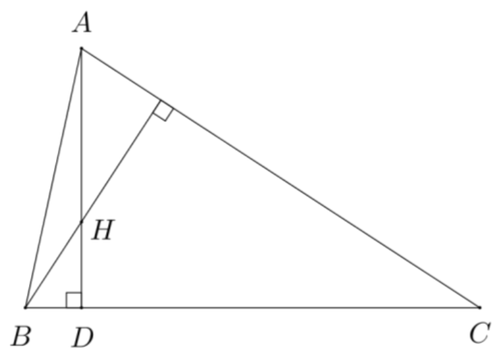
4. Let $x - y = d$. Setting $x = y + d$ in the given equation and simplifying, we get $(3d - 36)y^2 + (3d^2 - 36d)y + (2d^3 - 3456) = 0$. As $2d^3 - 3456 = 2(d^3 - 12^3)$, we can take out the common factor $d - 12$ to get $(d-12)[3y^2 + 3dy + 2(d^2 + 12d + 144)] = 0$. The second factor is a quadratic polynomial in y with discriminant

$$(3d)^2 - 4 \times 3 \times 2(d^2 + 12d + 144) = -15d^2 - 288d - 3456 = -6d^2 - 9(d+16)^2 - 1152 < 0.$$

Hence the quadratic factor can never be equal to 0, meaning that the only solution is $d = 12$.

5. A positive integer is divisible by 36 if and only if it is both a multiple of 4 and 9. Since the last two digits are equal and they form multiple of 4, there are 3 possibilities (namely, 00, 44 and 88). There are 10 choices for each of the hundreds and the thousands digits. Finally, for any choice of the last four digits, there is exactly one choice of the first digit (out of 1 to 9, since the first digit cannot be 0) which makes the sum of digits of the resulting integer (and hence the integer itself) divisible by 9. It follows that the answer is $3 \times 10 \times 10 = 300$.

6. As $\Delta HBD \sim \Delta CAD$ (because $\angle HDB = \angle CDA = 90°$ and $\angle HBD$ = 90° – $\angle ACB$ = $\angle CAD$), we have $\frac{HD}{BD} = \frac{CD}{AD}$, which implies $BD \times CD = 3(3+4) = 21$. Together with $BD + CD = 12$, the lengths BD and CD are the roots of the equation $x^2 - 12x + 21 = 0$. From $AB < AC$, BD is the smaller root, i.e. $BD = \frac{12 - \sqrt{12^2 - 4 \times 21}}{2} = 6 - \sqrt{15}$.



7. Let $P(x)=x^3+2x^2+3x+4$. Note that $P(x)=(x-\alpha)(x-\beta)(x-\gamma)$. Note also that

$$\alpha^4-1=(1-\alpha^2)(-1-\alpha^2)=(1-\alpha)(-1-\alpha)(i-\alpha)(-i-\alpha) .$$

Similarly, we have

$$\beta^4-1=(1-\beta)(-1-\beta)(i-\beta)(-i-\beta)$$
$$\gamma^4-1=(1-\gamma)(-1-\gamma)(i-\gamma)(-i-\gamma)$$

Multiplying the three equations together, we have

$$(\alpha^4-1)(\beta^4-1)(\gamma^4-1)=P(1)P(-1)P(i)P(-i)=10\times 2\times(2+2i)\times(2-2i)=160 .$$

Remark. The answer can also be found in the 'traditional way' by using the facts that $\alpha+\beta+\gamma=-2$, $\alpha\beta+\beta\gamma+\gamma\alpha=3$ and $\alpha\beta\gamma=-4$, but that will involve more tedious computations.

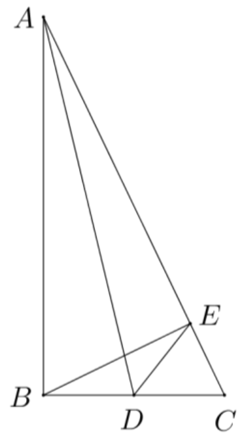
8. Pick any one boy and one girl. There are $3\times 2=6$ ways to assign the colours of their hats. Without loss of generality, suppose the boy is assigned a red hat and the girl is assigned a yellow hat. Now we try to assign hat to the remaining participants.

- If at least one boy has a blue hat, then all girls have yellow hats and so there are $2^7-1=127$ ways of assigning hats to the remaining boys (each may get a red or blue hat, but not all red).
- Similarly, if at least one girl has a blue hat, there are $2^9-1=511$ possibilities.
- If nobody is assigned a blue hat, there is only 1 possibility (all boys have red hats and all girls have yellow hats).

Hence the answer is $6\times(127+511+1)=3834$.

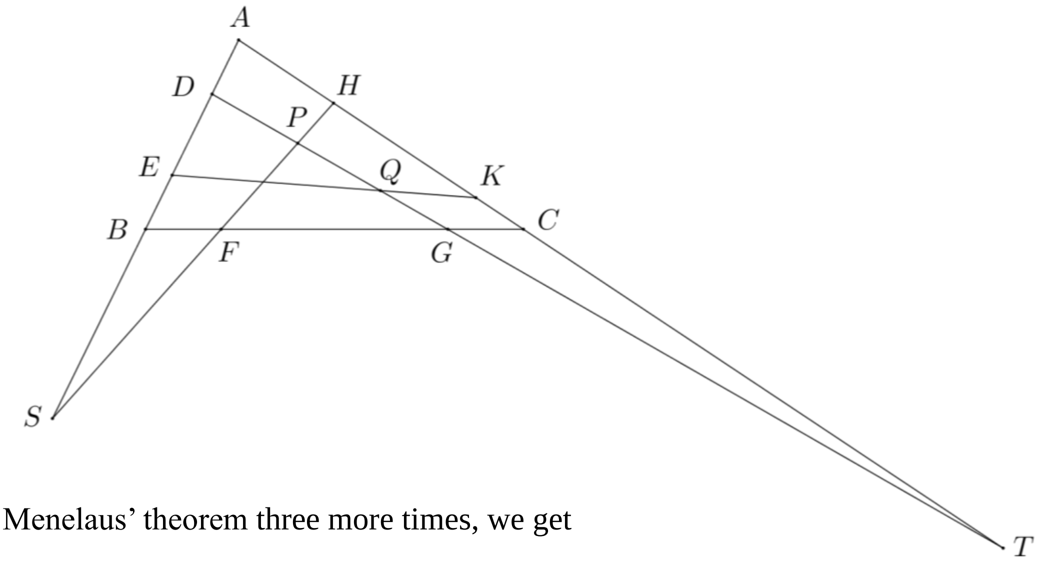
9. Let the three roots be $a-d$, a and $a+d$. Considering the sum of roots, we have $3a=-6$. Hence $a=-2$, which is one of the roots to the equation. Setting $x=-2$ in the equation thus gives $(-2)^3+6(-2)^2+5(-2)+k=0$, which implies $k=-6$. The equation in the question thus becomes $x^3+6x^2+5x-6=0$, or $(x+2)(x^2+4x-3)=0$. The roots are $x=-2$ and $x=-2\pm\sqrt{7}$. Hence the largest root is $\sqrt{7}-2$.

10. Let $AD = x$. Then we have $AB = \sqrt{AD^2 - BD^2} = \sqrt{x^2 - 1}$ and $AC = \sqrt{AB^2 + BC^2} = \sqrt{x^2 + 3}$. By the angle bisector theorem, we have $\frac{AE}{CE} = \frac{AD}{CD} = x$. Hence $AE = \frac{x}{x+1} AC = \frac{x\sqrt{x^2+3}}{x+1}$.



Consider the circle Γ with BC as diameter. Since $DB = DC = DE$, Γ has centre D and passes through E. Furthermore, AB is a tangent to Γ since $\angle ABD = 90°$. By the power chord theorem, we have $AB^2 = AE \times AC$, i.e. $x^2 - 1 = \frac{x(x^2+3)}{(x+1)}$. This simplifies to $x^2 - 4x - 1 = 0$, and the only positive solution is $x = 2 + \sqrt{5}$.

11. Extend AB meet HF to meet at S, and extend AC and DG to meet at T. By Menelaus' Theorem, we have $\frac{AS}{SB} \cdot \frac{BF}{FC} \cdot \frac{CH}{HA} = 1$ and $\frac{CT}{TA} \cdot \frac{AD}{DB} \cdot \frac{BG}{GC} = 1$. These give $\frac{AS}{SB} = 2$ and $\frac{CT}{TA} = \frac{5}{8}$ respectively, and we get $AD : DE : EB : BS = 2 : 3 : 2 : 7$ and $AH : HK : KC : CT = 2 : 3 : 1 : 10$.



Applying Menelaus' theorem three more times, we get

- $\frac{DP}{PT} \cdot \frac{TH}{HA} \cdot \frac{AS}{SD} = 1$, which gives $\frac{DP}{PT} = \frac{6}{49} = \frac{84}{686}$;
- $\frac{DQ}{QT} \cdot \frac{TK}{KA} \cdot \frac{AE}{ED} = 1$, which gives $\frac{DQ}{QT} = \frac{3}{11} = \frac{165}{605}$; and
- $\frac{DG}{GT} \cdot \frac{TC}{CA} \cdot \frac{AB}{BD} = 1$, which gives $\frac{DG}{GT} = \frac{3}{7} = \frac{231}{539}$.

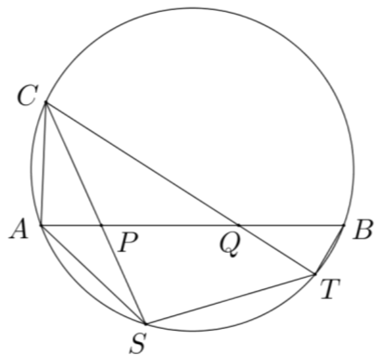
Note that the fractions have been expanded so that the sum of numerator and denominator of each fraction is 770 (which is the L.C.M. of $6+49=55$, $3+11=14$ and $3+7=10$). Hence if $DT = 770k$, then we have $DP = 84k$ and $QG = DG - DQ = 231k - 165k = 66k$. It follows that $DP : QG = 14 : 11$.

Remark. There are many alternative solutions to this problem. An obvious one is to use coordinate geometry (although quite an amount of computation will be involved). One may also use concepts from 'mass point' techniques. For example, to find $DP:PG$, one may assign a mass of 20 to A and split the masses at B and C. Using the ratios $AD:DB$ and $AH:HC$, the partial masses at B and C towards A will be 8 and 10 respectively. One can then use the ratio $BF:FG:GC$ to work out the partial masses at B towards C and C towards B to be 3.2 and 2.8 respectively. Now considering DG, the masses at D and G will be 28 and 16 respectively, which shows that $DP:PG=4:7$. One can then find $DQ:QG$ in a similar way.

12. There must be either 1 or 3 odd numbers in each row as well as each column in order for all row sums and column sums to be odd. Since there are 5 odd numbers in total, there must be a row containing 3 odd numbers. The other two rows each contains one odd number, and these two odd numbers must be in the same column in order for all column sums to be odd.

    In other words, there are 9 ways to choose the cells to place the odd numbers (3 choices for the row to contain 3 odd numbers, and then 3 choices for the column to place the remaining 2 odd numbers). As far as the required probability is concerned, only the parities of the numbers in each cell, rather than the numbers themselves, are relevant. As the number of ways of choosing 5 cells for the odd numbers is $C_5^9=126$, the required probability is $\dfrac{9}{126}=\dfrac{1}{14}$.

13. Let R denote the radius of the circle. By the extended sine formula, we have $\dfrac{5}{\sin\angle ACS}=\dfrac{AS}{\sin\angle ACS}=2R=\dfrac{ST}{\sin\angle TCS}$, i.e. $ST=5\times\dfrac{\sin\angle TCS}{\sin\angle ACS}$. Using $[XYZ]$ to denote the area of ΔXYZ, we have



$$\frac{5}{2}=\frac{QP}{AP}=\frac{[QCP]}{[ACP]}=\frac{\frac{1}{2}\cdot QC\cdot CP\cdot\sin\angle TCS}{\frac{1}{2}\cdot AC\cdot CP\cdot\sin\angle ACS}=\frac{QC\sin\angle TCS}{AC\sin\angle ACS}$$

Note that $\dfrac{QC}{AC}=\dfrac{QB}{TB}=\dfrac{4}{2}=2$ since $\Delta QCA\sim\Delta QBT$. Hence we have $\dfrac{\sin\angle TCS}{\sin\angle ACS}=\dfrac{5}{4}$ and so $ST=5\times\dfrac{5}{4}=\dfrac{25}{4}$.

14. Let $y = x + 2 + \sqrt{x^2 + 4x + 3}$. Note that $y \neq 0$ (for otherwise $\sqrt{x^2 + 4x + 3} = -x - 2$, and squaring both sides gives $x^2 + 4x + 3 = x^2 + 4x + 4$ which is impossible). We have

$$\frac{1}{y} = \frac{1}{x + 2 + \sqrt{x^2 + 4x + 3}} \cdot \frac{x + 2 - \sqrt{x^2 + 4x + 3}}{x + 2 - \sqrt{x^2 + 4x + 3}} = x + 2 - \sqrt{x^2 + 4x + 3} .$$

Hence the equation in the question becomes $y^5 - \frac{32}{y^5} = 31$, i.e. $(y^5 - 32)(y^5 + 1) = 0$. It follows that *y* may be 2 or –1.

- If $x + 2 + \sqrt{x^2 + 4x + 3} = 2$, we have $x^2 + 4x + 3 = (-x)^2$, which gives $x = -\frac{3}{4}$.
- If $x + 2 + \sqrt{x^2 + 4x + 3} = -1$, we have $x^2 + 4x + 3 = (-x - 3)^2$, which gives $x = -3$.

We check that both solutions satisfy the original equation. The answer is thus $\left(-\frac{3}{4}\right) \times (-3) = \frac{9}{4}$.

15. Direct checking shows that values of *n* which do not work include 2, 3, 4, 5, 7, 11, 13, … We shall prove below that all primes are not possible values of *n*, while all composite numbers greater than 4 are possible values of *n*.

- We first show that prime numbers do not work. If *p* is prime, then the power of *p* in the prime factorisation of $(p^2 - 1)!$ is $p - 1$ (since there are $p - 1$ multiples of *p* from 1 to $p^2 - 1$) while that in the prime factorisation of $(p!)^p$ is *p*. It follows that $(p^2 - 1)!$ is not a multiple of $(p!)^p$.
- Now let *n* be a composite number greater than 4. Note that $\frac{(n^2 - 1)!}{(n!)^n} = \frac{(n^2)!}{(n!)^{n+1}} \cdot \frac{(n-1)!}{n}$, and we shall prove that each of $\frac{(n^2)!}{(n!)^{n+1}}$ and $\frac{(n-1)!}{n}$ is an integer. Indeed, the former is always an integer (for all positive integers *n*) since it is the number of ways of dividing n^2 students into *n* groups of equal size (permute the students in $(n^2)!$ ways, let the first *n* students form a group, the next *n* students form another group, and so on, but within each group we divide by $n!$ as the permutation of the students does not affect the grouping, and finally divide by $n!$ as there are $n!$ ways to permute the groups). As for the latter, *n* must divide $(n-1)!$ because:
  - if $n = ab$ for distinct integers $a, b > 1$, then *a* and *b* are distinct terms in $(n-1)!$;
  - otherwise $n = p^2$ for some odd prime *p*, so *p* and $2p$ are distinct terms in $(n-1)!$.

This completes the proof of the claim at the beginning of the solution. Among 1 to 100 we have to take away the 25 primes as well as the number 4. The answer is thus $100 - 25 - 1 = 74$.

16. Since f is of degree 3, at most three of $f(2)$, $f(3)$, $f(4)$, $f(6)$, $f(7)$ and $f(8)$ can be equal to 16 (otherwise the polynomial $g(x)=f(x)-16$, which is also of degree 3, would have more than three zeros, contradicting the Fundamental Theorem of Algebra). Similarly at most three of them can be equal to -16. Hence we must have $f(a)=f(b)=f(c)=16$ and $f(p)=f(q)=f(r)=-16$, where $\{a,b,c,p,q,r\}=\{2,3,4,6,7,8\}$. By the remainder theorem, we have $f(x)=k(x-a)(x-b)(x-c)+16$ for some constant k (which is the leading coefficient of f) and similarly $f(x)=k(x-p)(x-q)(x-r)-16$. Hence we have

$$k(x-a)(x-b)(x-c)+16=k(x-p)(x-q)(x-r)-16$$

Comparing the coefficients of x^2 gives $a+b+c=p+q+r$, while comparing the constant terms gives $32=k(abc-pqr)$. As $2+3+4+6+7+8=30$, we either have $\{a,b,c\}=\{2,6,7\}$ and $\{p,q,r\}=\{3,4,8\}$, or the other way round, from the former equation. The latter equation then gives $k=-\frac{8}{3}$, or $k=\frac{8}{3}$ in the case where $\{a,b,c\}$ and $\{p,q,r\}$ are swapped. In either case we have

$$|f(0)|=\left|-\frac{8}{3}(-2)(-6)(-7)+16\right|=240.$$

17. Draw the triangles one by one — the first triangle divides the plane into 2 regions, and the second, third, fourth, … triangle would add at most 6, 12, 18, … regions respectively. In general, when we draw the k-th triangle, at most $6(k-1)$ new regions can be created (because each of the 3 sides of the new triangle may intersect each of the $k-1$ existing triangles, and each such intersection creates at most 2 more regions). Hence, after 100 triangles have been drawn, the number of regions is at most $2+(6+12+18+\cdots+6\times 99)=29702$.

Remark. The maximum value can be attained as long as every new triangle intersects every existing triangle at 6 points and no three sides are concurrent. This can be achieved by, for example, drawing an arbitrary triangle and rotating it by a small angle 99 times to create 100 triangles, so that any two triangles intersect at 6 points. Whenever three sides are concurrent, we move one of the triangles involved a little bit to remove the concurrency.

18. Let $t=xyz$. Then $t^3=(xyz)^3=(xyz-16)(xyz+3)(xyz+40)=t^3+27t^2-568t-1920$. Solving gives $t=24$ or $t=-\frac{80}{27}$. In the former case we have $(x,y,z)=(2,3,4)$ and $x^2+y^2+z^2=29$, while in the latter case we have $(x,y,z)=\left(-\frac{8}{3},\frac{1}{3},\frac{10}{3}\right)$ and $x^2+y^2+z^2=\frac{55}{3}$. One can check both satisfy the equations. Hence the smallest possible value of $x^2+y^2+z^2$ is $\frac{55}{3}$.

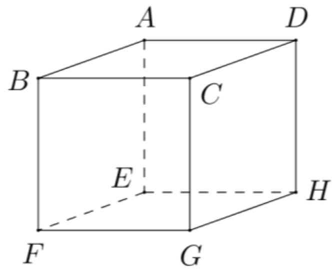
19. Using the half-angle formula $\tan\frac{\theta}{2}=\frac{1-\cos\theta}{\sin\theta}=\frac{\sin\theta}{1+\cos\theta}$, the sine formula and the cosine formula, we have (with standard notations in trigonometry),

$$\frac{2}{3}=\tan\frac{A}{2}\tan\frac{C}{2}=\frac{1-\cos A}{\sin A}\cdot\frac{\sin C}{1+\cos C}=\frac{1-\cos A}{1+\cos C}\cdot\frac{\sin C}{\sin A}=\frac{1-\frac{b^2+c^2-a^2}{2bc}}{1+\frac{a^2+b^2-c^2}{2ab}}\cdot\frac{c}{a}$$

$$=\frac{-(b^2-2bc+c^2)+a^2}{(a^2+2ab+b^2)-c^2}=\frac{(a-b+c)(a+b-c)}{(a+b-c)(a+b+c)}=\frac{a-b+c}{a+b+c}=\frac{17-6+c}{17+6+c}$$

Solving gives $c=13$.

20. Consider the centres of the eight faces of the regular octahedron. If we connect two adjacent faces by a line segment, we would get a cube, which we denote by *ABCDHGFE* as shown in the figure. The problem then becomes assigning the numbers 4, 5, …, 11 to the vertices of the cube and finding the probability that the numbers assigned to each pair of adjacent vertices are relatively prime.



We start with the number 6, and without loss of generality assume it is assigned to *A*. Among the remaining numbers, only 5, 7 and 11 are relatively prime to 6. Hence, in order to meet the condition of the question, the numbers assigned to *B*, *D* and *E* must be 5, 7 and 11 in some order. The probability for this to happen is $\frac{1}{C_3^7}=\frac{1}{35}$.

We must then allocate the numbers 4, 8, 9, 10 to *C*, *F*, *G*, *H*, and to meet the condition of the question 9 must be assigned to *G* (for otherwise there would be two adjacent even numbers), the probability of which is $\frac{1}{4}$.

Finally, 4, 8 and 10 are to be allocated to *C*, *F*, *H*, and the only restriction is that 10 must not be assigned to two vertices adjacent to the one assigned 5. The probability for this is $\frac{1}{3}$.

It follows that the answer is $\frac{1}{35}\times\frac{1}{4}\times\frac{1}{3}=\frac{1}{420}$.