International Mathematical Olympiad

Preliminary Selection Contest 2017 — Hong Kong

Outline of Solutions

Answers:

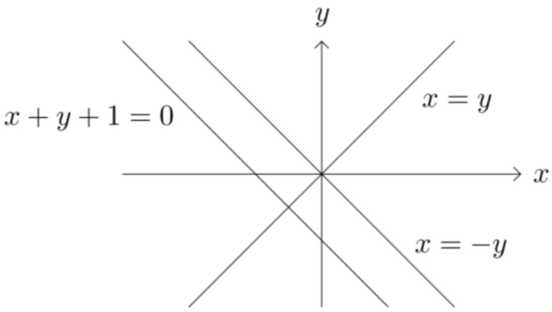
1. 1	**2.** 12	**3.** $\frac{2016}{2017}$	**4.** 30000
5. 3*	**6.** 597	**7.** 26	**8.** –7007
9. $\frac{64}{5}$	**10.** $\frac{15+6\sqrt{2}}{4}$	**11.** $\frac{23}{128}$	**12.** $\frac{3\sqrt{77}}{616}$
13. $\frac{37}{5}$	**14.** 17	**15.** 30	**16.** 37
17. $\sqrt{26}$	**18.** 315	**19.** 384	**20.** $\frac{2015}{8}$

*See the remark after the solution.

Solutions:

1. We consider the remainders when a_0, a_1, a_2, … are divided by 7. Note that when we compute the remainder when a_n is divided by 7, it suffices to replace a_{n-2} and a_{n-1} by the respective remainders in the equation $a_n = a_{n-2} + (a_{n-1})^2$ (e.g. once we know $a_3 = 5$ and $a_4 = 27 \equiv 6$ (mod 7), then we have $a_5 = a_3 + a_4{}^2 \equiv 5 + 6^2 \equiv 6$ (mod 7)). Thus it is easy to find that the remainders are respectively 1, 2, 5, 6, 6, 0, 6, 1, 0, 1, 1, 2, 5, 6, …, which repeat every 10 terms. The remainder when a_{2017} is divided by 7 is therefore the same as that when a_7 is divided by 7, which is 1 from the above list.

2. From $x^2(x+y+1) = y^2(x+y+1)$, we have $x^2 = y^2$ or $x+y+1=0$. The former is the same as $x = y$ or $x = -y$. Each of these equations represents a straight line. Therefore, we can draw the figure below. In particular, the lines $x+y+1=0$ and $x=-y$ are parallel and hence they have no intersection. One easily counts that there are 12 regions in total.



3. Note that for each integer n greater than 1, the expression for $f(n)$ consists of 2016 terms. We consider the contribution from each of these terms. Let S_k be the contribution from the term $\frac{1}{k^n}$ where $2 \le k \le 2017$. Then

$$S_k = \frac{1}{k^2} + \frac{1}{k^3} + \frac{1}{k^4} + \cdots = \frac{\frac{1}{k^2}}{1 - \frac{1}{k}} = \frac{1}{k^2 - k} = \frac{1}{k-1} - \frac{1}{k}.$$

It follows that

$$\begin{aligned} f(2) + f(3) + f(4) + \cdots &= S_2 + S_3 + \cdots + S_{2017} \\ &= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{2016} - \frac{1}{2017}\right) \\ &= 1 - \frac{1}{2017} \\ &= \frac{2016}{2017} \end{aligned}$$

4. We need to count the number of positive integer solutions to the equation $a + b + c = 600$ such that $a \le b \le c$. Then $1 \le a \le 200$, and we note that

- each even a (say, $a = 2k$) leads to $301 - 3k$ solutions (e.g. when $a = 100$, there are 151 solutions with $(b, c) = (100, 400)$, (101, 399), …, (250,250);
- each odd a (say, $a = 2k - 1$) leads to $302 - 3k$ solutions (e.g. when $a = 99$, there are 152 solutions with $(b, c) = (99, 402)$, (100, 401), …, (250, 251).

Thus the answer is thus $\sum_{k=1}^{100} (301 - 3k) + (302 - 3k) = 603 \cdot 100 - 6 \cdot \frac{100 \cdot 101}{2} = 30000$.

5. Clearly $n > 1$. If $n = 2$, we may let $x_1 = \frac{a}{b}$ and $x_2 = \frac{c}{d}$ where a, b, c, d are positive integers. Then $x_1^{\,3} + x_2^{\,3} = 1$ implies $(ad)^3 + (bc)^3 = (bd)^3$, contradicting Fermat's Last Theorem (which says that when n is an integer greater than 2 the equation $x^n + y^n = z^n$ has no positive integer

solution). Finally, as $3^3+4^3+5^3=6^3$, we have $\left(\frac{3}{6}\right)^3+\left(\frac{4}{6}\right)^3+\left(\frac{5}{6}\right)^3=1$ and so $n=3$ is possible. It follows that the answer is 3.

Remark. In the live paper, the condition 'less than 1' was accidentally missing. That would make the problem trivial with answer 1. Both 1 and 3 were accepted as correct during the contest.

6. First note that *b* cannot be 1, so there are 199 possible values for *b*. Now the equation can be rewritten as $\left(\frac{\log a}{\log b}\right)^{2017}=\frac{2017\log a}{\log b}$, i.e. $(\log a)^{2017}=2017(\log a)(\log b)^{2016}$. If $\log a=0$, which means $a=1$, then any of the 199 values of *b* would work. If $\log a\neq 0$, we can simplify the equation as $\log a=\pm\sqrt[2016]{2017(\log b)^{2016}}$. This equation has two solutions in *a* for each of the 199 possibilities for *b*. Hence the total number of solutions is $199+199\times 2=597$.

7. As $30=2\times3\times5$ and $3000=2^3\times3\times5^3$, each of *x*, *y*, *z* is of the form $2^a\times3\times5^b$, where each of *a*, *b* is 1, 2 or 3. Furthermore, among the three *a*'s chosen, one of them must be 1 and one of them must be 3, leading to 12 choices for the three *a*'s (including 6 permutations of (1,2,3), 3 permutations of (1,1,3) and 3 permutations of (1,3,3)). By the same argument there are 12 choices for the three *b*'s, leading to a total of $12\times12=144$ choices.

   However, because of the requirement $x\le y\le z$, many of these have to be discarded. In most cases, 1 out of 6 will work because of the permutations of the values of *x*, *y* and *z*. In some cases two of *x*, *y*, *z* are equal (note that *x*, *y*, *z* cannot be all equal), leading to only 3 permutations. There are 4 sets of (x,y,z) for which two of *x*, *y*, *z* are equal namely, $(x,z)=(2^1\times3\times5^1,2^3\times3\times5^3)$ and $(x,z)=(2^3\times3\times5^1,2^1\times3\times5^3)$, with *y* being equal to either *x* or *z*. Hence, among the 144 choices mentioned in the previous paragraph, the number of choices satisfying $x\le y\le z$ is

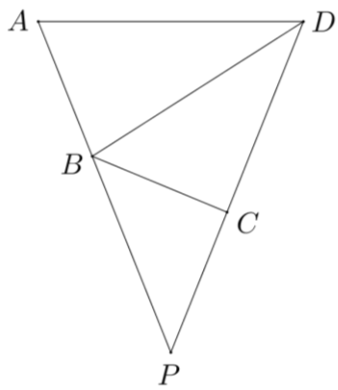
$$4+\frac{144-3\times4}{6}=26.$$

8. The condition implies $f(x)=(x-k)g(x)$ for some constant *k*. This gives

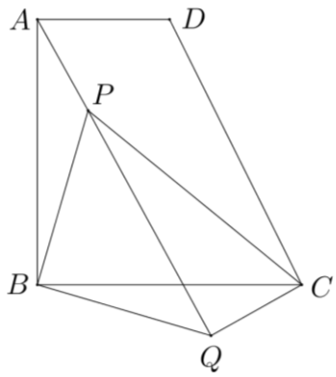
$$x^4+x^3+bx^2+100x+c=x^4+(a-k)x^3+(1-ak)x^2+(10-k)x-10k.$$

   By comparing the coefficient of *x*, we get $k=-90$. This implies $a=k+1=-89$. Hence $f(1)=(1-k)g(1)=(1-k)(a+12)=-7007$.

9. Suppose the extension of AB and CD meet at P. From $\angle ABD = \angle BCD$, we find that $\Delta PBC \sim \Delta PDB$. Then $\frac{PB}{PD} = \frac{BC}{DB} = \frac{3}{5}$. As $PB = PA - AB = PD - 8$, we obtain $PD = 20$. Using the similar triangles again, we have $\frac{PC}{PB} = \frac{3}{5}$. This implies $PC = \frac{36}{5}$. Hence $CD = 20 - \frac{36}{5} = \frac{64}{5}$.



10. Rotate P about B by 90° clockwise to obtain point Q. Then ΔPBQ is right-angled and isosceles. From $BA = BC$, $BP = BQ$ and $\angle ABP = 90^\circ - \angle PBC = \angle CBQ$, we have $\Delta ABP \cong \Delta CBQ$. This implies $CQ = AP = 1$. Also, we have $PQ^2 = BP^2 + BQ^2 = 2^2 + 2^2 = 8$. As $CQ^2 + PQ^2 = 9 = PC^2$, ΔPQC is right-angled at Q. Hence we have $\angle APB = \angle CQB = 45^\circ + 90^\circ = 135^\circ$, and so it follows that $AB^2 = 1^2 + 2^2 - 2(1)(2)\cos 135^\circ = 5 + 2\sqrt{2}$. The area of $ABCD$ is thus $\frac{(AD+BC)\times AB}{2} = \frac{3}{4}AB^2 = \frac{15+6\sqrt{2}}{4}$.



11. If Ann is to win, then one the following cases must happen.

- If the first 4 votes all go to Ann, she wins. The probability for this to happen is $\frac{1}{2^4} = \frac{1}{16}$.
- Suppose exactly 3 out of the first 4 votes are given to Ann (with probability $C_3^4 \times \frac{1}{2^4} = \frac{1}{4}$).
  - Ann wins if the next 2 votes are both go to her, with probability $\frac{1}{4} \times \frac{1}{2^2} = \frac{1}{16}$.
  - If she gets exactly 1 vote among the next 2, she has to get both of the remaining votes to win, with probability $\frac{1}{4} \times \left(C_1^2 \times \frac{1}{2^2} \right) \times \frac{1}{2^2} = \frac{1}{32}$.
- Suppose exactly 2 out of the first 4 votes are given to Ann (with probability $C_2^4 \times \frac{1}{2^4} = \frac{3}{8}$). Then the remaining 4 votes should all go to Ann in order that she can win. The probability for this to happen is $\frac{3}{8} \times \frac{1}{2^4} = \frac{3}{128}$.

The answer is thus $\frac{1}{16} + \frac{1}{16} + \frac{1}{32} + \frac{3}{128} = \frac{23}{128}$.

12. Without loss of generality, assume $d_1 < d_2 < \cdots < d_n$. Note that n is even and $d_j d_{n+1-j} = 11!$ for $1 \le j \le \frac{n}{2}$. For convenience, we write $m = \sqrt{11!}$. We find that

$$\begin{aligned}\frac{1}{d_j + m} + \frac{1}{d_{n+1-j} + m} &= \frac{d_j + d_{n+1-j} + 2m}{(d_j + m)(d_{n+1-j} + m)} \\ &= \frac{d_j + d_{n+1-j} + 2m}{m^2 + (d_j + d_{n+1-j})m + m^2} \\ &= \frac{1}{m}\end{aligned}$$

Hence the summands in the expression can form $\frac{n}{2}$ pairs so that the sum of each pair is $\frac{1}{m}$. As $11! = 2^8 \times 3^4 \times 5^2 \times 7 \times 11$, we have $n = (8+1)(4+1)(2+1)(1+1)(1+1) = 540$ and so the answer is $\frac{540}{2\sqrt{11!}} = \frac{3\sqrt{77}}{616}$.

13. If $k = 1$, we have $x = 6$; if $k = -1$, we have $x = -3$, which should be rejected.

When $k \ne \pm 1$, the equation is quadratic and can be factorised as $[(k+1)x - 12][(k-1)x - 6] = 0$. The solutions are $\frac{12}{k+1}$ and $\frac{6}{k-1}$. Let $\frac{12}{k+1} = m$ where m is a positive integer. Then $k = \frac{12}{m} - 1$. For $\frac{6}{k-1} = \frac{3m}{6-m}$ to be a positive integer, the denominator must be positive and so m is at most 5. Furthermore, we need $6 - m$ to divide 3m, and we check that only m = 3, 4, 5 work, and these corresponds to $k = 3$, 2 and $\frac{7}{5}$ respectively.

The answer is thus $1 + 3 + 2 + \frac{7}{5} = \frac{37}{5}$.

14. Note that two of the solutions come from $Q(x) = 1$ while the other two come from $Q(x) = -1$. Let c be a positive integer solution to $Q(x) = 1$. Then the other root is $-a - c$ from the sum of roots. Also, we have $c(-a - c) = b - 1$ from the product of roots. Similarly, let d be a root to $Q(x) = -1$. Then the other root is $-a - d$ and we have $d(-a - d) = b + 1$.

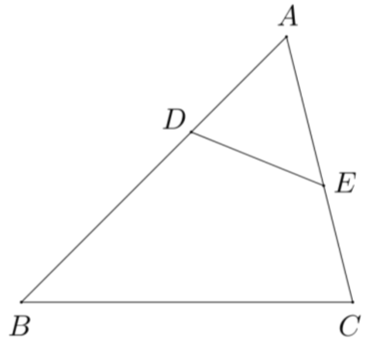
Using these to eliminate b, we obtain $d(-a - d) - c(-a - c) = 2$, which implies $(c - d)(a + c + d) = 2$. Note that c, d and $-a - c$ are positive integers. Hence, we have $(c - d, a + c + d) = (1, 2)$, (2,1), (–1,–2) or (–2,–1). This shows that a is odd. Also, from $-a - c > 0$, we know that a is positive.

Let $a = -(2m - 1)$ where $m \ge 1$. Since $c = \frac{(c-d) + (a+c+d) - a}{2}$, we check from the above four cases that c is either $m + 1$ (in the first two cases) or $m - 2$ (in the last two cases). In either case we have $b = c(-a - c) + 1 = m^2 - m - 1$, and it can be checked that the solutions to

$Q(x)^2 = 1$ are always $m-2$, $m-1$, m and $m+1$. Since these are positive integers, we must have $m \geq 3$. Also, from $b = m^2 - m - 1 \leq 365$, we get $m \leq 19$. It is easy to see that each of 3, 4, …., 19 is a possible value of *m*, and each corresponds to one set of (a,b). The answer is thus 17.

15. Applying cosine formula on ΔADE and ΔABC, we get

$$\frac{n^2+(21-n)^2-n^2}{2n(21-n)} = \cos A = \frac{33^2+21^2-m^2}{2(33)(21)}.$$



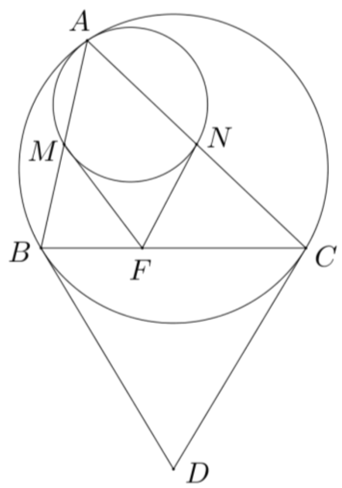
This simplifies to give $n(2223-m^2) = 21^2 \times 33 = 3^3 \times 7^2 \times 11$. From $n < AC = 21$ and $2n = AD + DE > AE = 21 - n$, we get $n = 9$ or 11.

When $n = 9$, $2223 - m^2 = 3 \times 7^2 \times 11$ has no integer solution. Thus we try $n = 11$, which gives $m = 30$ as the only possible solution.

16. Suppose there are *n* families in total and suppose there are *m* children. There are *m* choices for the best child, then $n-1$ choices for the best mother (since the best mother cannot be from the same family as the best child) and similarly $n-2$ choices for the best father.

Hence we have $m(n-1)(n-2) = 7770$. As $m \leq 5n$, we have $7770 \leq 5n(n-1)(n-2) < 5n^3$. This implies $n \geq 12$. Since $(n-1)(n-2)$ is a factor of $7770 = 2 \times 3 \times 5 \times 7 \times 37$, it is easy to check that the only possibility is $n = 16$, which corresponds to $m = 37$.

17. Consider the homothety with centre *A* and ratio $\frac{1}{2}$. Then points *B* and *C* are mapped to the midpoints *M* and *N* of *AB* and *AC* respectively. Let the tangents at *M* and *N* to the circumcircle of ΔAMN intersect at a point *F*. Then *F* is the image of *D* under the homothety, and so *F* lies on *BC* by the given condition that *A* and *D* are equidistant from *BC*.



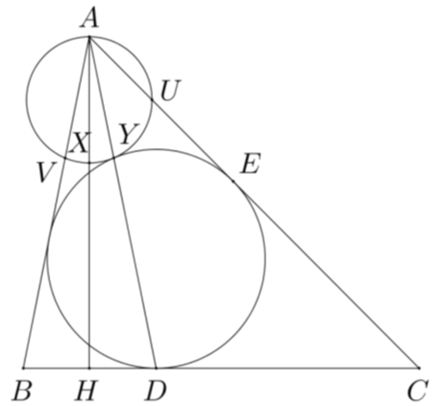
Note that $\angle BMF = \angle ANM = \angle ACB$ from tangent properties and the fact that *MN* and *BC* are parallel. This shows that *A*, *M*, *F*, *C* are concyclic. Hence we have $BF \times BC = BM \times BA = 8$. Similarly, we have $CF \times CB = CN \times CA = 18$. Adding these, we obtain $BC^2 = BF \times BC + CF \times CB = 26$. Thus $BC = \sqrt{26}$.

18. For each X_i, there are $C_2^4 = 6$ blue lines not passing through it. Hence, there are $5 \times 6 = 30$ red lines in total. They form $C_2^{30} = 435$ 'intersections' (including intersections of parallel lines and counting multiplicities when three or more lines meet at a point). However, the following should be discounted.

- For each X_i, there are 6 red lines passing through it. The $C_2^6 = 15$ intersections formed from these lines coincide. Hence, $5 \times (15-1) = 70$ intersections need to be discounted.
- For each of the $C_2^5 = 10$ blue lines, there are 3 red lines which are perpendicular to it. These 3 red lines are parallel and hence do not intersect. Thus $10 \times C_2^3 = 30$ intersections need to be discounted.
- The three altitudes of a triangle are concurrent. There are $C_3^5 = 10$ triangles formed from the 5 points. Hence another $10 \times 2 = 20$ intersections should be discounted.

The other intersections do not coincide in general. Hence, the maximum number of points of intersection is $435 - 70 - 30 - 20 = 315$.

19. Let *a*, *b*, *c* be the lengths of *BC*, *CA*, *AB* respectively. Let *r* and *s* be the inradius and semi-perimeter of ΔABC. By considering the area of ΔABC, we have $rs = \sqrt{s(s-a)(s-b)(s-c)}$. Hence we have



$$BD \times DC = (s-b)(s-c) = \frac{r^2 s}{s-a}.$$

Applying the extended sine law, we have $\frac{UV}{\sin A} = AX$. This gives $\sin A = \frac{4}{5}$. As the triangle is acute, this implies $r = AE \tan\frac{A}{2} = AE \cdot \frac{\sin A}{1+\cos A} = 12$.

Let *H* be the foot of altitude from *A* to *BC* and let *Y* be point of tangency of the two circles. Consider the homothety about *Y* that sends ω' to ω. As the tangent at *A* to ω' is parallel to the tangent at *D* to ω, the points *A*, *Y*, *D* are collinear. From $\angle XYD = \angle XHD = 90^\circ$, points *X*, *H*, *D*, *Y* are concyclic. It follows that $AX \times AH = AY \times AD = AE^2$. This gives $AH = \frac{24^2}{15} = \frac{192}{5}$, and from $rs = \frac{a}{2} \cdot AH$ we get $a = \frac{5s}{8}$. It follows that $BD \times DC = \frac{r^2 s}{s-a} = 12^2 \times \frac{8}{3} = 384$.

20. Suppose *n* is a 'good' number with respect to a set *X* with *k* elements. Then both *n* and $n+k$ belong to *X*, so we must have $k \geq 2$ and $k \leq 2017 - n$. The remaining $k-2$ elements can be chosen from the remaining 2015 elements. Thus there are C_{k-2}^{2015} such sets *X*, and this gives

$$C_0^{2015}+C_1^{2015}+C_2^{2015}+\cdots+C_{2015-n}^{2015}$$

sets for which n is ‘good’. Note that n is at most 2015. Summing over all n, the total number of ‘good’ positive integers is

$$S=2015C_0^{2015}+2014C_1^{2015}+2013C_2^{2015}+\cdots+C_{2014}^{2015}.$$

Using the relation $C_r^m=C_{m-r}^m$, we get

$$S=2015C_{2015}^{2015}+2014C_{2014}^{2015}+2013C_{2013}^{2015}+\cdots+C_1^{2015}.$$

Adding these, we obtain

$$2S=2015(C_0^{2015}+C_1^{2015}+C_2^{2015}+\cdots+C_{2015}^{2015})=2015\times 2^{2015}.$$

Hence, the required expected value is $\dfrac{S}{2^{2017}}=\dfrac{2015\times 2^{2014}}{2^{2017}}=\dfrac{2015}{8}$.