International Mathematical Olympiad

Preliminary Selection Contest 2015 — Hong Kong

Outline of Solutions

Answers:

1.	4	**2.**	667	**3.**	$6°$	**4.**	59
5.	432	**6.**	–54	**7.**	35	**8.**	$3\sqrt{55}$
9.	6	**10.**	2	**11.**	15	**12.**	6
13.	$\frac{3\sqrt{3}}{4}$	**14.**	1320	**15.**	4060224	**16.**	30
17.	906	**18.**	$\frac{121}{162}$	**19.**	$\frac{11}{9}$	**20.**	31185

Solutions:

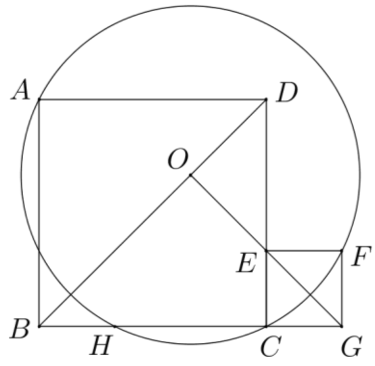
1. Suppose all lamps are turned on after n rounds. Then we have pressed the switches $5n$ times in total. Note that each lamp should change state for an odd number of times. As there are 12 lamps, the total number of times the lamps have changed state should be an even number. This forces n to be even.

   Clearly $n \neq 2$, since at most $5 \times 2 = 10$ lamps can be turned on in 2 rounds. On the other hand we can turn on all lamps in 4 rounds as follows:

   - Round 1 — Press switches 1, 2, 3, 4, 5
   - Round 2 — Press switches 6, 7, 8, 9, 10
   - Round 3 — Press switches 7, 8, 9, 10, 11
   - Round 4 — Press switches 7, 8, 9, 10, 12

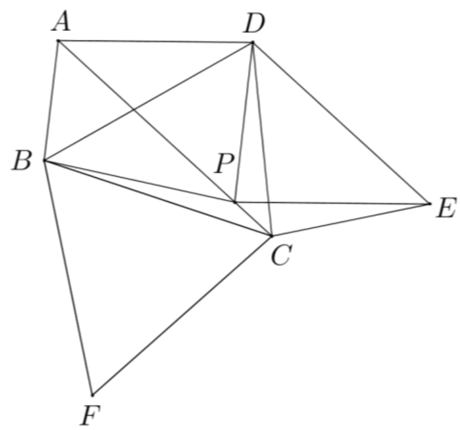
   It follows that the answer is 4.

2. Note that BD is the perpendicular bisector of AC, while EG is the perpendicular bisector of CF. Thus the intersection of BD and EG, which we denote by O, is the circumcentre of ΔACF.



As $\angle OBG = \angle OGB = 45°$, ΔOBG is isosceles. Since $OH = OC$, we have $BH = CG = 567$. It follows that $CH = 1234 - 567 = 667$.

3. As shown in the figure, let D be the point for which $ABPD$ is an isosceles trapezium with AB // DP. Let also E be the point for which $APED$ is a parallelogram, and F be the point such that A and F lie on different sides of BC and for which CBF is an equilateral triangle.



Note that by our construction and the given condition $AP = BC$, we have $DE = BD = BF$ as each of these three segments has the same length as AP.

We have $\angle CBD = \angle CBA - \angle DBA = (180° - 54° - 24°) - 54° = 48°$. Hence we get

$$\angle BDE = \angle BDP + \angle PDE = \angle BDP + \angle DPA = 54° + 54° = 108°$$

and

$$\angle FBD = \angle FBC + \angle CBD = 60° + 48° = 108°.$$

These show E, D, B, F are consecutive vertices of a regular pentagon. It follows that C and E both lie on the perpendicular bisector of BF. Thus we have $\angle DEC = \dfrac{108°}{2} = 54° = \angle PDE$.

Together with PC // DE, we see that $PCED$ is an isosceles trapezium. In particular, we have $CD = PE = AD$. Hence we have

$$\angle PAD = \angle DCA = \angle DCB - \angle ACB = \frac{180° - 48°}{2} - 24° = 42°$$

and so $\angle CBP = \angle CBA - \angle PBD - \angle DBA = 102° - \angle PAD - 54° = 6°$.

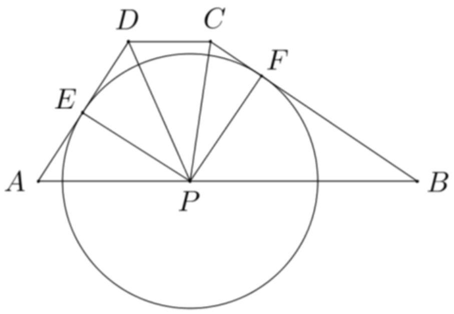
4. For convenience we define $f(1) = 1$ and $f(n+2) = (n+2)^{f(n)}$ for odd positive integers n. Then the question asks for the last two digits of $f(19)$.

Since $f(13) = 13^{f(11)}$ is odd, we have $f(15) = 15^{f(13)} \equiv (-1)^{f(13)} = -1 \equiv 3 \pmod 4$. As $f(17) = 17^{f(15)}$ and the unit digits of the powers of 7 (also powers of 17) follow the pattern 7, 9, 3, 1, 7, 9, 3, 1 which repeat every four terms, we conclude that the unit digit of $f(17)$ is the same as that of 17^3, which is 3.

Finally, if we look at the last two digits of the powers of 19, we will see the pattern 19, 61, 59, 21, 99, 81, 39, 41, 79, 01, 19, 61, which repeat every 10 terms. As $f(17)$ has unit digit 3, the last two digits of $f(19)=19^{f(17)}$ are the same as those of 19^3, which are 59.

Remark. When solving the problem one naturally proceeds in the opposite direction, i.e. look for patterns of the last two digits of the powers of 19 first.

5. Let r be the radius of the circle, and E, F be the points where the circle touches AD and BC respectively. Then $PE=PF=r$ and they are the heights of ΔPAD and ΔPBC from P.



Let h be the height of the trapezium. By considering the areas of ΔPAD and ΔPBC, we get $\dfrac{PA\cdot h}{2}=\dfrac{15r}{2}$ and $\dfrac{PB\cdot h}{2}=\dfrac{20r}{2}$. Thus $\dfrac{PA}{PB}=\dfrac{3}{4}$. Together with $AB=42$, we have $PA=18$ and $PB=24$. It follows that $PA\times PB=432$.

6. Since $a+b$ satisfies the given equation, we have $(a+b)^2+a(a+b)+b=0$. Rearranging gives $b^2+(3a+1)b+2a^2=0$. This means b is a root to the equation $x^2+(3a+1)x+2a^2=0$. As a and b are integers, the discriminant $(3a+1)^2-4\cdot 2a^2=a^2+6a+1$ must be a perfect square.

Let $a^2+6a+1=m^2$. Then $(a+3)^2-8=m^2$ and so $(a+3-m)(a+3+m)=8$. As the two terms $a+3-m$ and $a+3+m$ have the same parity (they differ by $2m$ which is even), they can only be 2 and 4, or –2 and –4 (up to permutation). The corresponding possible values of a are 0 and -6.

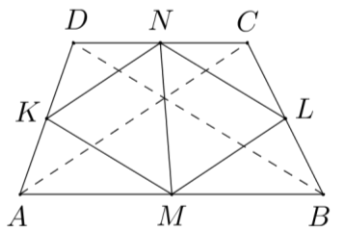
When $a=0$, the equation becomes $b^2+b=0$, giving $b=0$ or $b=-1$. When $a=-6$, the equation becomes $b^2-17b+72=0$, giving $b=8$ or $b=9$. The smallest possible value of ab is thus $(-6)\times 9=-54$.

7. Note that $f\left(\dfrac{1}{x}\right)=\dfrac{15}{\frac{1}{x}+1}+\dfrac{16}{\frac{1}{x^2}+1}-\dfrac{17}{\frac{1}{x^3}+1}=\dfrac{15x}{1+x}+\dfrac{16x^2}{1+x^2}-\dfrac{17x^3}{1+x^3}$. Hence we have

$$f(x)+f\left(\frac{1}{x}\right)=15\left(\frac{1}{x+1}+\frac{x}{x+1}\right)+16\left(\frac{1}{x^2+1}+\frac{x^2}{x^2+1}\right)-17\left(\frac{1}{x^3+1}+\frac{x^3}{x^3+1}\right)=15+16-17=14.$$

Since $\tan 15° = \dfrac{1}{\tan 75°}$, $\tan 30° = \dfrac{1}{\tan 60°}$ and $\tan 45° = 1$, the value of the expression in the question is $14 + 14 + f(1) = 14 + 14 + \dfrac{15}{2} + \dfrac{16}{2} - \dfrac{17}{2} = 35$.

8. We use $[XYZ]$ to denote the area of XYZ. Let K and L be the midpoints of AD and BC respectively. Then $\Delta DKN \sim \Delta DAC$ with side length ratio $1:2$ and so $[DKN] = \dfrac{1}{4}[DAC]$. Similarly, we have $[AMK] = \dfrac{1}{4}[ABD]$, $[BLM] = \dfrac{1}{4}[BCA]$ and $[CNL] = \dfrac{1}{4}[CDB]$. Summing up these relations, we get $[DKN] + [AMK] + [BLM] + [CNL] = \dfrac{1}{2}[ABCD]$.



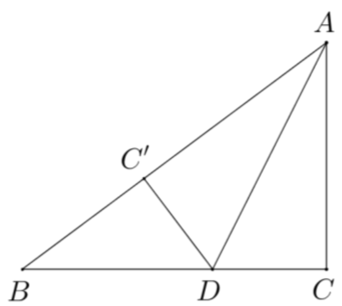
It follows that $[ABCD] = 2[KMLN] = 4[NML]$. By the mid-point theorem, we have $NL = \dfrac{1}{2}BD = 4 = NM$ and $ML = \dfrac{1}{2}AC = 3$. Hence the height of ΔNML from N has length $\sqrt{4^2 - \left(\dfrac{3}{2}\right)^2} = \dfrac{\sqrt{55}}{2}$. Thus $[NML] = \dfrac{3\sqrt{55}}{4}$ and so $[ABCD] = 3\sqrt{55}$.

9. Let m be an integral root to the equation. Then we have $km^2 + (4k-2)m + (4k-7) = 0$, which can be rewritten as $(m^2 + 4m + 4)k = 2m + 7$, or $(m+2)^2 k = 2m + 7$. Hence $m+2$ divides $2m+7$, and so $m+2$ divides $2m + 7 - 2(m+2) = 3$ as well. Thus m can only be –5, –3, –1 or 1, which we plug in to $(m+2)^2 k = 2m + 7$ one by one:

- When $m = -5$, we get $9k = -3$ which gives no integer solution for k.
- When $m = -3$, we get $k = 1$.
- When $m = -1$, we get $k = 5$.
- When $m = 1$, we get $9k = 9$ which again gives $k = 1$.

The sum of all possible values of k is thus $1 + 5 = 6$.

10. As AD is the internal bisector of $\angle A$, point C' lies on AB. Since $\Delta DBC' \sim \Delta ABC$, we have $\angle BC'D = \angle ACB$. Together with $\angle AC'D = \angle ACB$, we have $\angle BC'D = \angle AC'D$ and so each of $\angle BC'D$, $\angle AC'D$ and $\angle ACB$ is equal to 90°.



Let $BC = x$. Then $AC = \dfrac{2}{x}$ and $AB = \sqrt{x^2 + \dfrac{4}{x^2}}$. Using the AM-GM inequality $y + \dfrac{c}{y} \geq 2\sqrt{c}$ for positive numbers *y* and *c*, the perimeter of ΔABC is

$$x + \frac{2}{x} + \sqrt{x^2 + \frac{4}{x^2}} \geq 2\sqrt{2} + \sqrt{2\sqrt{4}} = 2 + 2\sqrt{2}.$$

This minimum value is obtained when $x = \dfrac{2}{x}$ and $x^2 = \dfrac{4}{x^2}$, i.e. when $x = \sqrt{2}$ (which corresponds to $AB = \sqrt{\sqrt{2}^2 + \dfrac{4}{\sqrt{2}^2}} = 2$).

11. Let *p* be any prime factor of $n+1$. Then *p* is a prime factor of $(n+1)(2n+15)$, hence of $n(n+5)$ as well. Since $n(n+5) = (n+1)(n+4) - 4$, we conclude that *p* divides 4 and so p can only be 2. In the same way, we can find that the only possible prime divisors of $n+5$ are 2 and 5.

Let $n+1 = 2^a$ and $n+5 = 2^b 5^c$. Then we have $2^a + 4 = 2^b 5^c$. Note that if $a \geq 5$, the left hand side is a multiple of 4 but not a multiple of 8. Hence we must have $b = 2$ and the equation becomes $2^{a-2} + 1 = 5^c$. As $a \geq 5$, this gives $5^c \equiv 1$ (mod 8), forcing c to be even. However, when c is even, we have $2^{a-2} = 5^c - 1 \equiv (-1)^c - 1 = 0$ (mod 3), which is impossible.

This means *a* can only be 0, 1, 2, 3 or 4. To find the greatest possible value of *n*, we set $a = 4$ to get $n = 15$. We can check that with this we have $(n+1)(2n+15) = 16 \times 45 = 2^4 \times 3^2 \times 5$ and $n(n+5) = 15 \times 20 = 2^2 \times 3 \times 5^2$, so 15 is indeed a possible value of *n*. It follows that the answer is 15.

12. Since a is a root to $g(x) = 0$, we have $a^4 - a^3 - a^2 - 1 = 0$. Using this relation, we get $f(a) = a^6 - a^5 - a^3 - a^2 - a = (a^2+1)(a^4 - a^3 - a^2 - 1) + a^2 - a + 1 = a^2 - a + 1$. This also applies to *b*, *c* and *d*, and hence $f(a) + f(b) + f(c) + f(d) = (a^2 + b^2 + c^2 + d^2) - (a+b+c+d) + 4$.

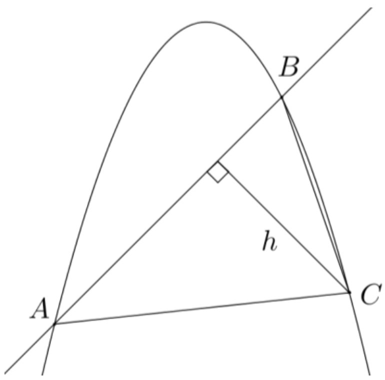
Since *a*, *b*, *c*, *d* are roots to the equation $x^4 - x^3 - x^2 - 1 = 0$, we have $a+b+c+d = 1$ and $ab + ac + ad + bc + bd + cd = -1$. It follows that

$$a^2 + b^2 + c^2 + d^2 = (a+b+c+d)^2 - 2(ab+ac+ad+bc+bd+cd) = 3$$

and so $f(a) + f(b) + f(c) + f(d) = 3 - 1 + 4 = 6$.

Remark. In the first step we were trying to 'simplify' $a^6 - a^5 - a^3 - a^2 - a$ subject to the constraint $a^4 - a^3 - a^2 - 1 = 0$. To do this we divided the former by the left hand side of the latter and ended up with $a^2 - a + 1$, which is in fact the remainder of the division.

13. Since A and B both lie on the straight line $y=x+k$, we may let $A=(\alpha,\alpha+k)$ and $B=(\beta,\beta+k)$. Combining the equations of the straight line and the parabola, we get $x^2+x+k-1=0$. Its two roots are α and β since the two graphs meet at A and B. Hence we have $\alpha+\beta=-1$ and $\alpha\beta=k-1$. It follows that



$$AB^2=2(\alpha-\beta)^2=2(\alpha+\beta)^2-8\alpha\beta=10-8k.$$

The height h from C to AB is $\frac{|1+k|}{\sqrt{2}}$. So the area of ΔABC is $\frac{AB\cdot h}{2}=\frac{1}{2}\sqrt{(5-4k)(1+k)^2}=\frac{1}{2}\sqrt{2\left(\frac{5}{2}-2k\right)(1+k)^2}$.

Finally, by the AM-GM inequality, we have

$$\left(\frac{5}{2}-2k\right)(1+k)^2\le\left[\frac{\left(\frac{5}{2}-2k\right)+(1+k)+(1+k)}{3}\right]^3=\frac{27}{8}.$$

Equality holds when $\frac{5}{2}-2k=1+k$, or $k=\frac{1}{2}$. The maximum possible area of ΔABC is thus $\frac{1}{2}\sqrt{2\cdot\frac{27}{8}}=\frac{3\sqrt{3}}{4}$.

14. Note that for any $1\le a,b\le 10$, the equation $ax\equiv b$ (mod 11) has exactly one solution (modulo 11). This can be proved by techniques in number theory, or one can just check case by case — for example when $a=3$, one can check that as x runs through 1 to 10, ax modulo 11 also run through 1 to 10 albeit in a different order (i.e. the remainders when 3, 6, 9, 12, 15, 18, 21, 24, 27, 30 are divided by 11 are respectively 3, 6, 9, 1, 4, 7, 10, 2, 5, 8). Hence the equation has a unique solution for any b such that $1\le b\le 10$.

Now let p, q, r, s be the numbers in the four boxes from left to right. If $q\ne 11$ (i.e. $q\not\equiv 0$ (mod 11), then regardless of the values of r and s, we can find a unique p which gives a correct answer as remarked above. As there are 10 choices for q and 11 choices for each of r and s, this leads to a total of $10\times 11\times 11=1210$ different correct answers.

When $q=11$, then $r=11$ would lead to no solution. Otherwise, when $r\ne 11$, then there is a unique choice for s no matter what p is, as remarked above. As there are 11 choices for p and 10 choices for r, there are $11\times 10=110$ different correct answers in this case.

Combining the two cases above, the answer is $1210+110=1320$.

15. We apply the formula $a^3+b^3=(a+b)(a^2-ab+b^2)$ to $a=10^{672}$ and $b=2015$ to get $10^{2016}=(10^{672}+2015)(10^{1344}-2015\cdot 10^{672}+2015^2)-2015^3$. From this, we have $\left[\dfrac{10^{2016}}{10^{672}+2015}\right]=10^{1344}-2015\cdot 10^{672}+2015^2+\left[-\dfrac{2015^3}{10^{672}+2015}\right]$. As 2015^3 is much smaller than $10^{672}+2015$, the term inside the square bracket lies between -1 and 0. Hence, the integer is equal to $10^{1344}-2015\cdot 10^{672}+2015^2-1$, whose last 7 digits are the same as those of 2015^2-1, which are 4060224.

16. If d is odd, then there is one even term among any two consecutive terms. Hence there are at least 5 even numbers in the sequence, so at least one of them is not equal to 2 or –2, and thus its absolute value is not prime. This shows that d must be even. The same argument shows that d must be divisible by 3.

Next we show that d is divisible by 5. If not, then there must be a multiple of 5 among any 5 consecutive terms, so there must be two multiples of 5 in the sequence. If the absolute value of each term of the sequence is prime, then these multiples of 5 must be –5 and 5. This is not possible, because then the four terms between them must be –3, –1, 1 and 3, contradicting the condition that the absolute value of each term must be prime.

It follows that d is at least 30. It is possible for d to be equal to 30. For example, we may take the arithmetic sequence –113, –83, –53, –23, 7, 37, 67, 97, 127, 157.

Remark. There is a mistake in the formulation of the question. It should ask for the minimum value of $|d|$ rather than that of d (otherwise d can be as negative as one desires), and it should specify that d is non-zero (otherwise the minimum value of $|d|$ must be zero since one can just take a sequence in which every term is equal to the same prime number).

17. The 12 students can be divided into four groups of 3, each forming an equilateral triangle. For each group, there are $2^3-2=6$ possible hat colours, thus making up a total of $6^4=1296$ colourings. However we need to subtract those cases in which there exist four students whose positions form a square are put on hats of the same colour. (The only possible regular polygons formed are equilateral triangles, squares, regular hexagons and the whole regular 12-gon. Once we settle the cases for the first two, i.e. ensuring no monochromatic equilateral triangle and monochromatic square, the last two cases need not be dealt with because there must be no monochromatic regular hexagon and monochromatic regular 12-gon.)

So, among the 1296 colourings, how many are there in which there is at least one monochromatic square? Again, the 12 students can be divided into three groups of 4, each forming a square.

- There are $2\times 3^4=162$ colourings in which one particular square is monochromatic.

(There are 2 choices to fix a colour for the monochromatic square, say, red. The remaining 8 vertices can be paired up according to the original ‘triangle grouping’ at the beginning, each allowing 3 colour combinations as we cannot have both red.)

- Now we count the number of colourings in which two particular squares are monochromatic. If these two monochromatic squares are of the same colour, all remaining vertices must be of the other colour. If they are of different colours, then each of the remaining vertices can be either red or blue. Hence there are $2+2\times 2^4=34$ colour combinations in this case.
- Finally, there are $2^3-2=6$ colour combinations in which all three squares are monochromatic. (Note that they cannot all be of the same colour since we are restricting ourselves to the 1296 colourings in which there is no monochromatic triangle.)

By inclusion-exclusion principle, $3\times 162-3\times 34+6=390$ of the 1296 colourings have at least one monochromatic square. It follows that the answer is $1296-390=906$.

18. Define $g(x)=f(3x)-f(x)-1$. Then $g(1)=\mathrm{g}(3)=\mathrm{g}(9)=\mathrm{g}(27)=\mathrm{g}(81)=0$. As $g(x)$ is a polynomial with degree at most 5, we have $g(x)=k(x-1)(x-3)(x-9)(x-27)(x-81)$ for some constant k. Since $g(0)=-1$, we have $k=\frac{1}{1\times 3\times 9\times 27\times 81}$. In other words, we have $g(x)=\frac{(x-1)(x-3)(x-9)(x-27)(x-81)}{1\times 3\times 9\times 27\times 81}$ and the coefficient of x is $\frac{1}{81}+\frac{1}{27}+\frac{1}{9}+\frac{1}{3}+\frac{1}{1}$.

Note that if the coefficient of x in $f(x)$ is c, then the same coefficient in $f(3x)$ would be $3c$ and hence that in $g(x)$ would be $2c$. In other words, the coefficient of x in $f(x)$ is half of that in $g(x)$. Therefore the answer is $\frac{1}{2}\left(\frac{1}{81}+\frac{1}{27}+\frac{1}{9}+\frac{1}{3}+\frac{1}{1}\right)=\frac{121}{162}$.

Remark. Clearly a solution using brute force exists but it is extremely tedious.

19. Let $b_n=\sqrt{24a_n+9}$. Then we have $a_n=\frac{b_n^2-9}{24}$, and the original recurrence relation becomes $\frac{b_{n+1}^2-9}{24}=\frac{b_n^2-9}{96}+\frac{b_n}{16}-\frac{9}{48}$, or $(2b_{n+1})^2=(b_n+3)^2$. As b_n is non-negative for all n, this yields $2b_{n+1}=b_n+3$. Rewrite this as $2(b_{n+1}-3)=b_n-3$. Let $c_n=b_n-3$. Then we have $c_{n+1}=\frac{1}{2}c_n$ and $c_1=b_1-3=\sqrt{24a_1+9}-3=2$. It follows that $c_n=\frac{1}{2^{n-2}}$, $b_n=\frac{1}{2^{n-2}}+3$ and $a_n=\frac{1}{24}\cdot\frac{1}{2^{2n-4}}+\frac{1}{2^n}$,

and so $a_1+a_2+a_3+\cdots=\frac{1}{24}\left(4+1+\frac{1}{4}+\cdots\right)+\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots\right)=\frac{1}{24}\cdot\frac{4}{1-\frac{1}{4}}+\frac{\frac{1}{2}}{1-\frac{1}{2}}=\frac{11}{9}$.

Remark. The constant $\frac{9}{48}$ in the question should have been simplified to $\frac{3}{16}$.

20. We first divide the 8 men into 4 groups of two. Pick any man A and there are 7 ways to choose his groupmate B. Then pick any other man C and there are 5 ways to choose his groupmate D. Next pick any one of the remaining men E and there are 3 ways to choose his groupmate F. Finally the last two men G and H must be in the same group. Hence there are $7\times 5\times 3=105$ ways to group the men. Now we number the groups 1 to 4 (i.e. A and B are in Group 1, and so on), and let a, b, …, h be the wives of A, B, …, H respectively. We consider two cases.

<u>Case 1: The husbands of the two women in Group 1 are in the same group</u>

There are 3 ways to choose two women into Group 1. WLOG assume c and d are put into Group 1. Then the following groupings are possible according to the positions of a and b:

- If a and b join the same group (3 possibilities), the grouping is fixed. (For example, if both a and b join Group 3, then since g and h cannot join Group 4 they are both forced to join Group 2.) Hence there are 3 arrangements in this case.
- If a joins Group 3 and b joins Group 4 (or vice versa, leading to 2 possibilities), there are 2 choices for each of the remaining place in Group 3 (for g or h) and Group 4 (for e or f), giving a total of $2\times 2\times 2=8$ arrangements in this case.
- Otherwise, one of a and b joins Group 2, the other joins either Group 3 or Group 4 (4 possibilities). Let's say a joins Group 2 and b joins Group 3. Then e and f must join Group 4 but g and h are free with 2 choices, giving $4\times 2=8$ arrangements in this case.

Altogether, there are $3\times(3+8+8)=57$ arrangements in Case 1.

<u>Case 2: The husbands of the two women in Group 1 are in different groups</u>

There are $C_2^6-3=12$ ways to choose two women into Group 1. WLOG assume c and e are put into Group 1. Then the following groupings are possible according to the positions of d and f:

- If d and f are in the same group, they must be in Group 4. Then a, b, g, h are free to join Group 2 and Group 3, so there are $C_2^4=6$ possible arrangements.
- If neither d nor f go to Group 4, then d and f must be in Group 3 and Group 2 respectively. Hence there are only 2 arrangements, with a and b joining Group 4 and g and h taking up the remaining two places in either way.
- Otherwise there are 2 ways for one of d and f to go to Group 4, say, d joins Group 4. Then f must be in Group 2. There are 3 possible arrangements for g and h (both join Group 3, or choose one of them to join Group 3 and the other to join Group 2), and then 2 possible arrangements for a and b. Hence there are totally $2\times 3\times 2=12$ arrangements in this case.

Altogether, there are $12\times(6+2+12)=240$ arrangements in Case 2.

Combining Cases 1 and 2, the answer is $105\times(57+240)=31185$.